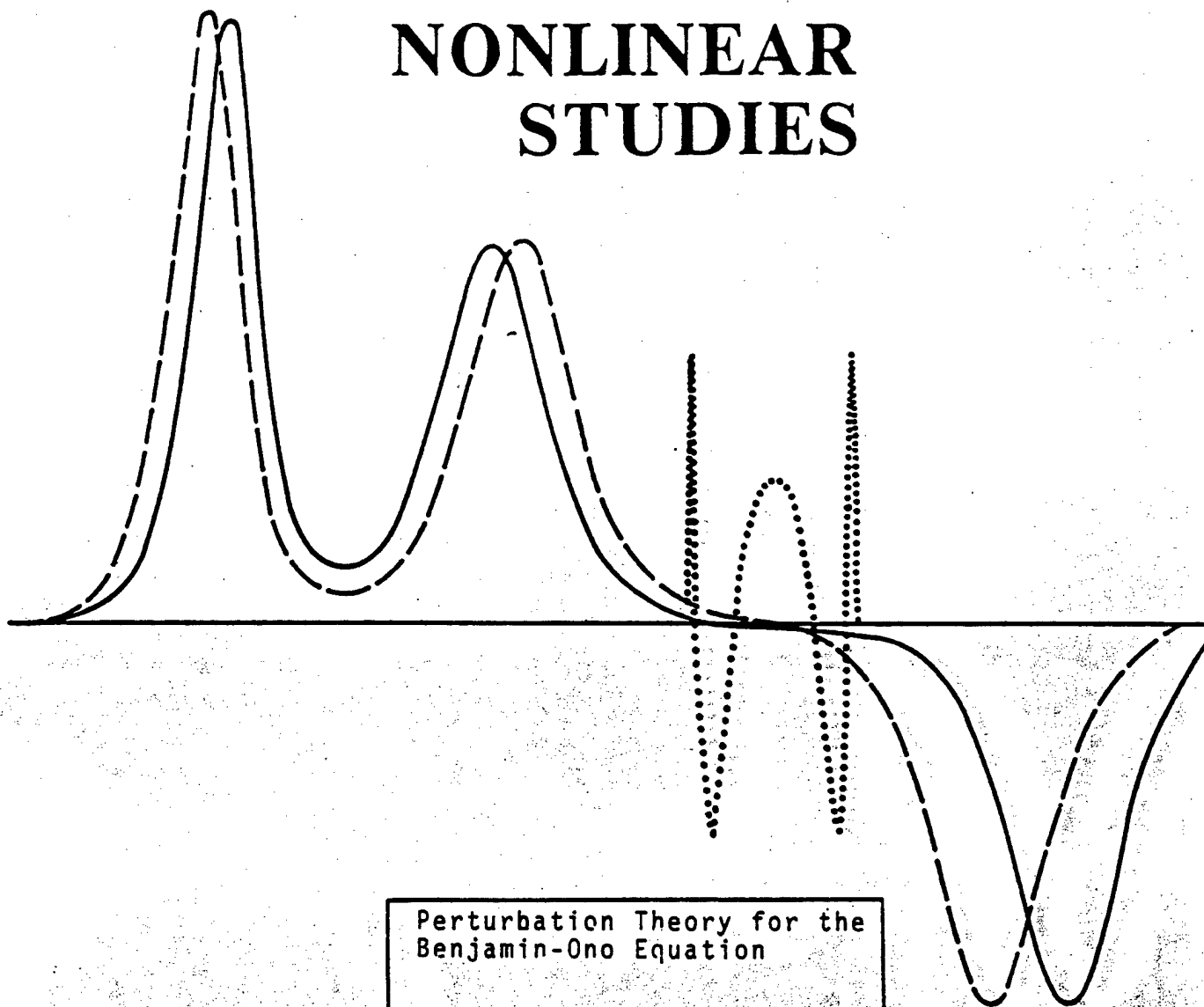# INSTITUTE
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Benjamin-Ono Equation

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Perturbation Theory for the Benjamin-Ono Equation

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Abstract

We develop a perturbation theory for the Benjamin-Ono (BO) equation. This perturbation theory is based on the Inverse Scattering Transform for the BO equation, which was originally developed by Fokas and Ablowitz and recently refined by Kaup and Matsuno. We find the expressions for the variations of the scattering data with respect to the potential, as well as the dual expression for the variation of the potential in terms of the variations of the scattering data. This allows us to introduce the squared eigenfunctions for the BO equation, whose completeness and orthogonality in both x- and λ-spaces we also establish. We consider the two most important applications of the developed machinery. First, we present an explicit first-order solution of the BO equation driven by a small perturbation. Second, we introduce the Poisson bracket and a set of the canonical action-angle variables for the BO equation. and thus demonstrate its complete integrability as a Hamiltonian dynamical system.

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1 Introduction

In this paper, we develop a perturbation theory for the Benjamin-Ono (BO) equation [1]:

$$u_t + 2uu_x + Hu_{xx} = 0 \qquad Hv(x) \equiv \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(\xi)}{\xi - x} d\xi \qquad (1.1)$$

where P stands for a principal value, and the background solution $u(x)$ is assumed to vanish at infinity. We also assume that $u(x)$ and its variation, $\delta u(x)$, are real functions, and that they and all their derivatives are smooth and decay sufficiently rapidly as $|x| \to \infty$. This problem has previously been addressed in a number of studies. As early as in 1980, Chen and Kaup [2] have found a general solution of the BO linearized on the background of an exact 1-soliton solution. Later on, Matsuno has developed a multi-soliton perturbation theory [3] which, however, did not allow one to compute the radiation emitted by the soliton. Very recently, Matsuno and Kaup have found [4] a general solution of the BO linearized about an N-soliton solution, thus generalizing the result of [2] and overcoming the limitations of [3]. The common feature of all these works was that the solution of the linearized BO was found without an explicit recourse to the exact integrability of the BO by the Inverse Scattering Transform (IST).

In the present paper, we develop the perturbation theory for the BO based on the IST formalism, which was developed in Ref. [5]. According to [5], $u(x)$ is the potential in the scattering problem for the corresponding operator in the Lax pair. We will also use refined information about the scattering data and their asymptotics as $\lambda \to 0$, that was recently obtained in Refs. [6, 7]. Of course, for any initial N-soliton solution, the results of our perturbation theory will agree completely with those of Ref. [4], obtained by algebraic means.

The plan of this paper is as follows. After a summary of relevant results from earlier works [5, 7, 8], presented in Section 2, we obtain the variational derivatives of the scattering data:

$$\frac{\delta \beta}{\delta u} \qquad \frac{\delta \lambda_j}{\delta u} \qquad \frac{\delta \gamma_j}{\delta u} \qquad (1.2)$$

where $\beta(\lambda)$ is the reflection coefficient, λ_j is the discrete eigenvalue, and γ_i is the normalization constant. This is done in Section 3. Then, in Section 4, we obtain the equation for $\delta u(x)$ and deduce from it the completeness relation for the "squared eigenfunctions" (SE). Also in Section 4, we produce an expansion of the potential $u(x)$ over the complete set of SE and demonstrate that the SE solve the direct and adjoint linearized BO. In Section 5, we first present, and then give a direct proof of, the orthogonality relation satisfied by the SE. In Section 6, we present the explicit first-order solution for the BO with an arbitrary perturbation, and then further specify the form of that solution, when the background solution of Eq. (1.1) is a single soliton. In Section 7, we introduce the Poisson brackets for the BO equation and hence prove its integrability as a Hamiltonian system. Finally, in Section 8, we summarize the results obtained and discuss the similarities of the perturbation theory for the BO with the perturbation theories for other integrable $1 + 1$ and $2 + 1$ evolution equations.

2 Background

The x-operator of the Lax pair for the BO is (see [5] and references therein):

$$i\phi_x^+ + \lambda(\phi^+ - \phi^-) = -u\phi^+. \qquad (2.1)$$

Here ϕ^+ and ϕ^- are functions analytic in the upper and lower halves of the complex x-plane (in what follows referred to as x-UHP and x-LHP), respectively, and $u(x)$ is the solution of (1.1). There are two solutions of (2.1), specified by their asymptotics at, say, $x \to +\infty$:

$$N(x, \lambda) \to e^{i\lambda x} \qquad x \to +\infty \qquad (2.2)$$

$$\overline{N}(x,\lambda) \to 1 \qquad x \to +\infty \tag{2.3}$$

They both are analytic functions in the x-UHP (i.e. they correspond to σ^- in (2.1)). These functions can be shown [5] to satisfy the following respective homogeneous and nonhomogeneous equations:

$$N_x - i\lambda N = iP^-(uN) \qquad (\lambda > 0 \text{ and real}) \tag{2.4}$$

$$\overline{N}_x - i\lambda\overline{N} = iP^-(u\overline{N}) - i\lambda \qquad (\lambda \text{ arbitrary}) \tag{2.5}$$

where $P^\pm$ are the projection operators defined by

$$P^\pm(v) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{v(y)}{y-(x\pm i\epsilon)}dy \qquad \epsilon = +0.$$

For $\lambda > 0$, they also satisfy [5] Fredholm integral equations of the same kernel, but with different inhomogeneous terms:

$$N(x,\lambda) = e^{i\lambda x} + \int_{-\infty}^{\infty} G_-(x,y,\lambda)u(y)N(y,\lambda)dy \tag{2.6}$$

$$\overline{N}(x,\lambda) = 1 + \int_{-\infty}^{\infty} G_-(x,y,\lambda)u(y)\overline{N}(y,\lambda)dy \tag{2.7}$$

where

$$G_-(x,y,\lambda) = G(x,y,\lambda - i\epsilon) \qquad G(x,y,\lambda) = \frac{1}{2\pi}\int_0^\infty \frac{e^{i(x-y)p}}{p-\lambda}dp. \tag{2.8}$$

Obviously, the function $G(x,y,\lambda)$, as a function of λ, is analytic in the *entire* λ-plane except for the positive semi-axis $\lambda > 0$. Therefore, in [8] we were able to define a function $W(x,\lambda)$ satisfying the Fredholm equation:

$$W(x,\lambda) = 1 + \int_{-\infty}^{\infty} G(x,y,\lambda)u(y)W(y,\lambda)dy \qquad (\lambda \notin [0,\infty)). \tag{2.9}$$

Since $G(x,y,\lambda)$ is analytic in λ away from $\lambda > 0$, then by Fredholm theory, the function $W(x,\lambda)$ is analytic in the entire λ-plane except for (i) the positive semi-axis and (ii) isolated points where homogeneous solutions of (2.9) exist. Then, $\overline{N}(x,\lambda)$ is the boundary value of this function from below: $\overline{N}(x,\lambda > 0) = W(x,\lambda - i\epsilon)$. In fact, in [5] there was defined and used yet another solution of (2.5), $M(x,\lambda)$, which was the boundary value of $W(x,\lambda)$ from above: $M(x,\lambda > 0) = W(x,\lambda + i\epsilon)$ (cf. also Fig. 1). Obviously, $M(x,\lambda)$ satisfies the equation of the form of (2.7), with G_- being replaced by $G_+(x,y,\lambda) = G(x,y,\lambda + i\epsilon)$. Since $G(x,y,\lambda)$ is analytic in x in the x-UHP, then $M(x,\lambda)$ is also analytic in that domain.

From the definitions of M and $\overline{N}$, the following relation can be easily obtained [5]:

$$M = \overline{N} + \beta(\lambda)N \qquad (\lambda > 0) \tag{2.10}$$

where $\beta(\lambda)$ is the reflection coefficient given by

$$\beta(\lambda) = i\int_{-\infty}^{\infty} u(y)M(y,\lambda)e^{-i\lambda y}dy. \tag{2.11}$$

In (2.10) and below, we will omit the arguments of functions when they are obvious. From Eqs. (2.2), (2.3), and (2.10) one finds the asymptotics of M as $x \to +\infty$:

$$M \to 1 + \beta e^{i\lambda x} \qquad (x \to +\infty \quad \lambda > 0). \tag{2.12}$$

The asymptotics of N, $\overline{N}$, and M at $x \to -\infty$ are [5, 7]:

$$N \to \Gamma^{-1}(\lambda)e^{i\lambda x} \qquad x \to -\infty \tag{2.13}$$

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$$\bar{N} \to 1 - \mathcal{I}(\lambda)\Gamma^{-1}(\lambda)e^{i\lambda x} \qquad x \to -\infty \qquad (2.14)$$

$$M \to 1 \qquad x \to -\infty \qquad (2.15)$$

where

$$\Gamma(\lambda) = \exp\left(\frac{1}{2\pi i}\int_0^\lambda \frac{\mathcal{I}(\lambda')\mathcal{I}^*(\lambda')d\lambda'}{\lambda'}\right). \qquad (2.16)$$

In Section 3 we will need asymptotics for N when $x \to \infty$ and x is in the UHP. This follows from the integral equation (2.6) and the asymptotic form of $G(x, y, \lambda)$ as $x \to \infty$:

$$G(x, y, \lambda) = \frac{1}{2\pi i x \lambda} + O\left(\frac{1}{x^2}\right). \qquad (2.17)$$

Substituting Eq. (2.17) into Eq. (2.6) and using Eqs. (2.34) and (2.35) below, one obtains:

$$N(x, \lambda) \to \bar{\Gamma}(\lambda)e^{i\lambda x} + \frac{\mathcal{I}^*(\lambda)}{2\pi x \lambda} + O\left(\frac{1}{x^2}\right) \qquad \lambda > 0. \qquad (2.18)$$

Here $\bar{\Gamma}(\lambda) = 1$ for $x \to +\infty$, $\bar{\Gamma}(\lambda) = \Gamma^{-1}(\lambda)$ for $x \to -\infty$, and for $\mathrm{Im}\,x \to +\infty$ the value of $\bar{\Gamma}$ is unimportant since the corresponding term in (2.18) is exponentially small.

We will also need the asymptotics of the Jost functions W and N as $\lambda \to \infty$. For W, this follows from the asymptotic form of G: $G = O(\frac{1}{\lambda})$, whence [5]

$$W \to 1 + O\left(\frac{1}{\lambda}\right) \qquad \lambda \to \infty. \qquad (2.19)$$

For N, the asymptotics for $\lambda > 0$ and real, and *for real x only*, can be obtained from (2.4) in the following way. As will be shown later, $N(x, \lambda \to \infty) \simeq e^{i\lambda x}N_\infty(x)$ for some $N_\infty(x)$. Substituting this form in (2.4) and using the well-known formula (which is valid only for real x and y) $\lim_{\lambda\to\infty}\frac{e^{i\lambda(y-x)}}{y-(x+i\epsilon)} = 2\pi i\delta(x-y)$, and also the asymptotics (2.2), one can determine this $N_\infty(x)$, finding:

$$N \to \exp\left(i\lambda x - i\int_x^\infty u(y)dy\right) \qquad x \text{ real} \quad \lambda \to +\infty. \qquad (2.20)$$

This relation was derived in [9]; however, it was not pointed out there that it is valid, in general, only for real x.

It was shown in [5] that the homogeneous solutions of (2.9) (i. e. those satisfying (2.9) without "1" in the rhs), which are also analytic function of x in the x-UHP, may exist for a finite number of isolated points $\lambda = \lambda_j$, $j = 1, \ldots$. Moreover, in [10] it was proven that all $\lambda_j < 0$. Each discrete eigenvalue λ_j in (2.1) gives rise to one soliton in the original evolution equation (1.1). The corresponding solution of (2.9), denoted as $\Phi_j(x)$, satisfies the equation:

$$\Phi_{j_x} - i\lambda_j\Phi_j = iP^+(u\Phi_j). \qquad (2.21)$$

The asymptotics of Φ_j is defined to be

$$\Phi_j(x) \to \frac{1}{x} \qquad x \to \infty. \qquad (2.22)$$

It was shown [5] that as $\lambda \to \lambda_j$, one has

$$W(x, \lambda) \to \frac{-i\Phi_j}{\lambda - \lambda_j} + (x + \gamma_j)\Phi_j + (\lambda - \lambda_j)\bar{W}_j(x) + O(\lambda - \lambda_j)^2 \qquad (2.23)$$

which *defines* the normalization constant γ_j. The exact form of $W_j(x)$ in Eq. (2.23) was not needed in earlier studies, so it will be determined in Section 3 below. The following relation [11] (see also [10, 7]) between γ_j and λ_j will be of future use in this paper:

$$\mathrm{Im}\gamma_j = -\frac{1}{2\lambda_j} > 0. \tag{2.24}$$

Thus, it is $\mathrm{Re}\gamma_j$ and λ_j that are the functionally independent discrete scattering data.

The following relations between $\overline{N}$ and N can be obtained [5]:

$$\overline{N}(\lambda) = 1 + \frac{1}{2\pi i}\int_0^\infty \frac{\beta(\lambda')N(\lambda')d\lambda'}{\lambda' - (\lambda - i\epsilon)} - i\sum_j \frac{\Phi_j}{\lambda - \lambda_j} \tag{2.25}$$

$$(x + \gamma_j)\Phi_j + i\sum_{k\neq j}\frac{\Phi_k}{\lambda_j - \lambda_k} - \frac{1}{2\pi i}\int_0^\infty \frac{\beta(\lambda')N(\lambda')d\lambda'}{\lambda' - \lambda_j} = 1. \tag{2.26}$$

Note that Eq. (2.25) can also be obtained by applying the Cauchy residue theorem to the function $\frac{W(\lambda')}{(\lambda' - (\lambda - i\epsilon))}$ and using Eqs. (2.10), (2.19), and (2.23). Then (2.26) is a consequence of (2.25) and (2.23).

To insure the convergence of the integral in (2.25), one needs to use the asympotics of the Jost functions and the reflection coefficient at $\lambda \to 0$ [7] (see also [5, 6]). These asymptotics are different in the generic and non-generic cases, with the latter including, but not being limited to, the pure N-soliton case. Following [7], define $N_{00}(x)$ as the solution of:

$$N_{00}(x) - i\int_x^\infty u(y)N_{00}(y)dy + \frac{1}{2\pi}\int_{-\infty}^\infty u(y)N_{00}(y)\ln(x - y - i\epsilon)dy = 1 \tag{2.27}$$

and then define I_{00} to be the following integral:

$$I_{00} = \int_{-\infty}^\infty u(x)N_{00}(x)dx. \tag{2.28}$$

Then in the generic case ($I_{00} \neq 0$), one has, as $\lambda \to 0$ [5, 6, 7]:

$$N(x, \lambda), \overline{N}(x, \lambda) \to \frac{2\pi N_{00}(x)}{I_{00}\ln\lambda} + O\left(\frac{1}{\ln^2\lambda}\right) \tag{2.29}$$

$$\beta(\lambda) \to \frac{2\pi i}{\ln\lambda} + O\left(\frac{1}{\ln^2\lambda}\right). \tag{2.30}$$

In the non-generic case ($I_{00} = 0$), one has [7]:

$$N(x, \lambda), \overline{N}(x, \lambda) \to N_{00}(x)[1 + O(\lambda\ln\lambda)] + O(\lambda) \tag{2.31}$$

$$\beta(\lambda) \to O(\lambda). \tag{2.32}$$

Also of great importance for the IST scheme of the BO equation is the connection formula between the Jost functions, established in [5]:

$$N_\lambda - ixN = f(\lambda)\overline{N} \tag{2.33}$$

where the subscript λ, as usual, means the corresponding partial differentiation, and

$$f(\lambda) = -\frac{1}{2\pi\lambda}\int_{-\infty}^\infty u(y)N(y, \lambda)dy \qquad \lambda > 0. \tag{2.34}$$

In [10] it was demonstrated that Eq. (2.33) is an analogue of the symmetry relation that usually exists between the Jost functions for (1+1) integrable models, when the potential in the Lax operator exhibits some symmetry or is, e.g., purely real (see, e.g., Eq. (4.22) or Eq. (4.27) in [12]). We emphasize, however,

that for the BO equation, the connection formula (2.33) was established without any assumption (except sufficient smoothness and decay properties) about the potential $u(x)$. Moreover, if one further assumes that $u(x)$ in Eq. (2.1) is real, then one can obtain [7] the following relation between $f(\lambda)$ and $\beta(\lambda)$:

$$f(\lambda) = \frac{\beta^*(\lambda)}{2\pi i\lambda}.$$ (2.35)

Note that in the limit $\lambda \to \infty$ it follows from (2.33) that

$$N(x,\lambda) = \exp(i\lambda x)\left[N(x,0) + \int_0^\infty f(\lambda)\overline{N}(x,\lambda')\exp(-i\lambda'x)d\lambda' - \int_\lambda^\infty f(\lambda)\overline{N}(x,\lambda')\exp(-i\lambda'x)d\lambda'\right].$$ (2.36)

For real x, the last term in Eq. (2.36) is exponentially small if $u(x)$ and all its derivatives are smooth and vanish at least as fast as $(1/x^2)$ at infinity (cf. (2.35), (2.11), (2.19)). Thus in the limit $\lambda \to +\infty$, one has $N(x,\lambda) \to \exp(i\lambda x)N_0(x)$, which was the starting point of the derivation of the asymptotics (2.20).

In what follows we will also require the orthogonality relations, first found in [7], between the Jost functions $N(x,\lambda)$ and $\Phi_j(x)$. Here we will present these relations in a simplified form, as pointed out in [8]:

$$\int_{-\infty}^\infty N^*(x,\lambda')N(x,\lambda)dx = 2\pi\delta(\lambda - \lambda')$$ (2.37)

$$\int_{-\infty}^\infty N^*(x,\lambda)\Phi_j(x)\,dx = 0 \qquad (\lambda > 0)$$ (2.38)

$$\int_{-\infty}^\infty \Phi_j^*(x)\Phi_k(x)\,dx = -2\pi\delta_{kj}\lambda_j$$ (2.39)

where δ_{kj} is the Kroneker's delta. Note that Eq. (2.38) holds in the sense of distributions, i.e. terms like $\lambda\delta(\lambda)$ have been neglected.

To conclude this summary of known results for the IST for the operator (2.1), we present here the completeness relation for the Jost functions, originally found in [8]:

$$\int_0^\infty d\lambda N^*(y,\lambda)N(x,\lambda) - \sum_j \frac{\Phi_j^*(y)\Phi_j(x)}{\lambda_j} = \frac{-i}{y - (x + i\varepsilon)}.$$ (2.40)

The rhs of Eq. (2.40) is equivalent to the conventional [13] term $2\pi\delta(x - y)$ when it acts on functions that are analytic in the x-UHP and decay as $x \to \infty$. In Appendix A, we outline an alternative derivation of Eq. (2.40), that is based on a purely algebraic procedure (it still uses, however, the IST equations (2.25), (2.26), and (2.33)).

Using Eqs. (2.37)–(2.40), one can write down the solution to an inhomogeneous form of Eqs. (2.4) or (2.21):

$$K_x(x,\lambda) - i\lambda K(x,\lambda) - iP^+(u(x)K(x,\lambda)) = R(x,\lambda)$$ (2.41)

where λ is either positive or is one of the λ_j for the given $u(x)$, and R is an arbitrary function analytic in the x-UHP. This equation naturally arises (see, e.g., [14]) when one considers the perturbed Jost functions, corresponding to the potential $u + \delta u$ (cf. Section 3 below). The general solution of Eq. (2.41) is:

$$K = CK_0 + \frac{1}{2\pi i}\left\{\int_0^\infty \frac{d\lambda' N(\lambda')\langle N^*(\lambda')R\rangle}{\lambda' - \lambda} - \sum_l \frac{\Phi_l\langle\Phi_l^*R\rangle}{\lambda_l(\lambda_l - \lambda)}\right\}$$ (2.42)

where the constant C is arbitrary, $K_0 = N(x,\lambda)$ for $\lambda > 0$ and $K_0 = \Phi_j(x)$ for $\lambda = \lambda_j$; in the latter case the sum in Eq. (2.42) does not contain the term with $l = j$. The notation $\langle\ldots\rangle$ here and below stands for $\int_{-\infty}^\infty dy(\ldots)$.

6

3 Variational Derivatives of the Scattering Data

Here we shall give the derivations of the variational derivatives (1.2). Consider the equation for $\delta M(x, \lambda)$ that follows from (2.5) upon replacing $u(x)$ with $u(x) + \delta u(x)$:

$$\delta M_x - i\lambda \delta M - iP^-(u\delta M) = iP^-(\delta u\, M). \tag{3.1}$$

Multiply Eq. (3.1) by $N^*(x, y)$ and integrate it over x from $-\infty$ to ∞, using Eq. (2.4), the asymptotics (2.2), (2.12), (2.13), and (2.15), and the relations

$$P^- N = N \qquad P^- N^* = N^*. \tag{3.2}$$

The result is:

$$\frac{\delta \mathcal{J}(\lambda)}{\delta u(x)} = i M(x, \lambda) N^*(x, \lambda). \tag{3.3}$$

To determine the other two quantities in the set (1.2), consider the analogue of Eq.(3.1) for δW. Multiply it by $\Phi_j^*(x)$ and integrate over x from $-\infty$ to ∞, with the result being:

$$(\lambda - \lambda_j)\langle \Phi_j^* \delta W \rangle = -\langle \delta u \Phi_j^* W \rangle. \tag{3.4}$$

In deriving (3.4), one needs to use Eq. (2.21) as well as the analyticity of $\Phi_j^*(x)$ in the x-LHP. Now use the expansion (2.23) of W near the pole $\lambda = \lambda_j$ and compare the consecutive powers of $(\lambda - \lambda_j)$ on both sides of Eq. (3.4). The term $O((\lambda - \lambda_j)^{-1})$ yields:

$$\frac{\delta \lambda_j}{\delta u} = \frac{1}{2\pi \lambda_j} \Phi_j^* \Phi_j \tag{3.5}$$

where we have used Eq. (2.39).

Before we proceed to the next order, $O(1)$, we need to obtain an equation for $\delta \Phi_j$, since this quantity appears in the pole expansion of δW. From the equation

$$(\delta \Phi_j)_x - i\lambda_j \delta \Phi_j - iP^+(u\delta \Phi_j) = i\delta\lambda_j \Phi_j + iP^+(\delta u\, \Phi_j), \tag{3.6}$$

and also Eqs. (3.5), (2.42), and (3.2) one easily obtains:

$$\delta \Phi_j(x) = C_j \Phi_j(x) + \frac{1}{2\pi} \int_0^\infty \frac{d\lambda}{\lambda - \lambda_j} N(x, \lambda)\langle \delta u \Phi_j N^* \rangle \tag{3.7}$$

$$- \frac{1}{2\pi} \sum_{l \neq j} \frac{\Phi_l(x)\langle \delta u \Phi_j \Phi_l^* \rangle}{\lambda_l(\lambda_l - \lambda_j)}.$$

Then substituting Eq. (3.7) into Eq. (3.4) and equating the coefficients at the $O(1)$ terms yields:

$$C_j = -\frac{1}{2\pi i \lambda_j} \langle (x + \gamma_j)\, \delta u \Phi_j^* \Phi_j \rangle. \tag{3.8}$$

We will now verify that this value of C_j guarantees that the asymptotics of $\delta \Phi_j$ at $x \to \infty$ is $O(|x|^{-2})$, as it is required by Eq. (2.22). Using Eq. (2.18), one can perform the integration in (3.7), obtaining:

$$\delta \Phi_j = \frac{1}{x}\left(C_j + \frac{\langle \delta u \Phi_j N^*(\lambda = 0)\rangle}{2\pi i \lambda_j} + \right. \tag{3.9}$$

$$\left. \frac{1}{(2\pi)^2} \int_0^\infty \frac{\beta^*(\lambda)\langle \delta u \Phi_j N^*(\lambda)\rangle d\lambda}{\lambda(\lambda - \lambda_j)} - \frac{1}{2\pi} \sum_{l \neq j} \frac{\langle \delta u \Phi_j \Phi_l^* \rangle}{\lambda_l(\lambda_l - \lambda_j)} \right) + O\left(\frac{1}{x^2}\right).$$

Next one uses the equation:

$$\overline{N}(x,0) = N(x,0)$$

(3.10)

(cf. Eqs. (2.29), (2.31)) and Eq. (2.25) to find $N^*(\lambda = 0)$, and then substitutes the result into Eq. (3.9). Finally, one uses Eqs. (3.8), (2.24), and the complex conjugate of Eq. (2.26) to verify that the $1/x$-term in Eq. (3.9) vanishes. This, along with the analyticity of $\delta\Phi_j$ in the x-UHP, implies that

$$\int_{-\infty}^{\infty} dx\, \delta\Phi_j = 0.$$

(3.11)

Now we can proceed to the next order, where $\delta\gamma_j, \delta u$ will be determined. To this end, we first refine the expansion (2.23) by specifying the form of $\bar{W}_j(x)$. This follows, for example, from Eq. (2.25):

$$\bar{W}_j(x) = \frac{1}{2\pi i} \int_0^{\infty} \frac{3(\lambda')N(x,\lambda')d\lambda'}{(\lambda' - \lambda_j)^2} - \sum_{l \neq j} \frac{i\Phi_l(x)}{(\lambda_l - \lambda_j)^2}.$$

(3.12)

We also note from Eq. (2.23) the following obvious relation that will be of future use (in Appendix D):

$$\bar{W}_j(x) = \frac{1}{2}\partial_\lambda^2[(\lambda - \lambda_j)W(x,\lambda)]_{\lambda=\lambda_j}.$$

(3.13)

Now, from Eqs. (3.4) and (2.23) we obtain in the order $O(\lambda - \lambda_j)$:

$$\left\langle \Phi_j^* \left[(x + \gamma_j)\delta\Phi_j + \delta\gamma_j\Phi_j - \delta\lambda_j\bar{W}_j \right] \right\rangle = -\left\langle \delta u\, \Phi_j^*\bar{W}_j \right\rangle.$$

(3.14)

Finally, we use the complex conjugate of Eq. (2.26) to compute $\langle\Phi_j^* x\delta\Phi_j\rangle$, and also Eqs. (3.7), (3.8), (2.37)–(2.39), (3.11), and (3.12) to obtain:

$$\frac{\delta\gamma_j}{\delta u} = -\frac{1}{2\pi\lambda_j^2}(x + \gamma_j)\,\Phi_j^*\Phi_j + i\sum_{l \neq j}\frac{\Phi_j^*\Phi_l - \Phi_j\Phi_l^*}{2\pi\lambda_j(\lambda_l - \lambda_j)^2}$$

(3.15)

$$+\frac{1}{(2\pi)^2 i\lambda_j}\int_0^{\infty}\frac{(\beta\Phi_j^*N - \beta^*\Phi_j N^*)d\lambda}{(\lambda - \lambda_j)^2}.$$

One can easily verify that Eq. (3.15) is consistent with Eqs. (2.24) and (3.5).

Remark: Note that $\lim_{|x|\to\infty}\delta\mathrm{Re}\gamma_j/\delta u(x) = 0$. This is in contrast to the case of the KdV, where the corresponding quantity has non-vanishing asymptotics (see Eq. (0.9) in [15]). This fact will be important in the calculation of the Poisson brackets between the variables of the continuous and discrete spectra in Section 7 below.

4 Variational Derivatives of the Potential and Completeness of the SE.

In this Section, we will first obtain the variation of the potential in terms of the variation of the scattering data and then present a completeness relation for the SE. Starting with Eq. (2.33) and its consequence for δN, one arrives at the following equation:

$$\partial_\lambda(N^*\delta N) = f\delta\overline{N}N^* + f^*\delta N\overline{N}^* + \delta f\, N^*\overline{N}.$$

(4.1)

Next, one uses Eqs. (2.10) and (2.35) to transform Eq. (4.1) into the form:

$$\partial_\lambda(N^*\delta N) = \frac{\delta\overline{N}M^* - \overline{N}^*\delta M}{2\pi i\lambda} + \frac{\delta\beta^*N^*\overline{N} + \delta\beta N\overline{N}^*}{2\pi i\lambda}.$$

(4.2)

Recall that N and M for $\lambda > 0$ are the boundary values $W(x, \lambda)$ from below and above, respectively. Then the first term in the r.h.s of Eq. (4.2) can be rewritten as:

$$\frac{1}{2\pi i \lambda}\, \delta W \cdot W^* \Big|_{\lambda - i\epsilon}^{\lambda - i\epsilon} \tag{4.3}$$

(cf. also Eq. (2.25) and its analogue for M, which differs from Eq. (2.25) by having $(\lambda + i\epsilon)$ instead of $(\lambda - i\epsilon)$). Then both sides of Eq. (4.2) can be integrated over λ from 0 to ∞, with the integration of the first term on the r.h.s. being performed by using the contour in Fig.1. We use the asymptotics (2.20), (2.29), (2.19) and Eqs. (2.23), (3.12), and (2.24) to obtain the final answer:

$$\delta u(x) = -\frac{\partial}{\partial x}\left\{ \sum_j \left[\frac{\Phi_j \Phi_j^*}{\lambda_j} \delta \mathrm{Re}\gamma_j - 2\pi \frac{\delta \mathrm{Re}\gamma_j}{\delta u(x)} \delta\lambda_j \right] \right. \tag{4.4}$$

$$\left. + \frac{1}{2\pi} \int_0^\infty \frac{(\delta\beta \overline{N}^* N + \delta\beta^* N^* \overline{N}) d\lambda}{\lambda} \right\},$$

where $(\delta\mathrm{Re}\gamma_j/\delta u)$ is found from Eq. (3.15). Using now the obvious identity $\delta u(x)/\delta u(y) = \delta(x - y)$, one can straightforwardly obtain from Eqs. (4.4), (3.3), (3.5), and (3.15) the completeness relation for the SE of the BO:

$$\delta(x - y) = \frac{\partial}{\partial x}\left\{ \sum_j \left[\frac{\Phi_j(y)\Phi_j^*(y)}{\lambda_j} \frac{\delta \mathrm{Re}\gamma_j}{\delta u(x)} - \frac{\Phi_j(x)\Phi_j^*(x)}{\lambda_j} \frac{\delta \mathrm{Re}\gamma_j}{\delta u(y)} \right] + \right. \tag{4.5}$$

$$\left. + \frac{1}{2\pi i} \int_0^\infty \frac{d\lambda}{\lambda} \left[N(x,\lambda)\overline{N}^*(x,\lambda)N^*(y,\lambda)M(y,\lambda) - N^*(x,\lambda)\overline{N}(x,\lambda)N(y,\lambda)M^*(y,\lambda) \right] \right\}.$$

In Appendix C, we outline a direct proof of Eq. (4.5) that uses the IST equations (2.25), (2.26), and (2.33). We note, however, that a proof of Eq. (4.5) that would use contour integration, which is well known for $1 + 1$ integrable equations (see, e.g., [16] or [17] and references therein), is still to be found for the BO.

Next, we will obtain the expansion of the background potential, $u(x)$, over the set of the SE. To that end, we first derive the following relations:

$$\int_{-\infty}^\infty u(x)\frac{\partial}{\partial x}\left(N^*(x,\lambda)\overline{N}(x,\lambda) \right) dx = -\lambda\beta(\lambda) \tag{4.6}$$

$$\int_{-\infty}^\infty u(x)\frac{\partial}{\partial x}\left(\Phi_j^*(x)\Phi_j(x) \right) dx = 0 \tag{4.7}$$

$$\int_{-\infty}^\infty u(x)\frac{\partial}{\partial x}\left(\frac{\delta\mathrm{Re}\gamma_j}{\delta u(x)} \right) dx = -1. \tag{4.8}$$

Eq. (4.6) can be obtained as follows. From Eqs. (2.4) and (2.5) one finds that

$$(N^*\overline{N})_x = \frac{1}{2}\left(\overline{N}H(uN^*) + N^*H(u\overline{N}) \right) - i\lambda N^* \tag{4.9}$$

(here H is the Hilbert operator, cf. Eq. (1.1)). Then Eq. (4.6) follows from Eq. (4.9), Eq. (5.15) below, and Eqs. (2.34) and (2.35). Eq. (4.7) is obtained similarly. To obtain, along the same lines, Eq. (4.8), a little more work is required. First one uses Eq. (D.1) in Appendix D to rewrite $\delta\mathrm{Re}\gamma_j/\delta u$ in terms of Φ_j and $W(\lambda \to \lambda_j)$. Next, from Eqs. (2.5) and (2.21) one obtains:

$$\partial_x(\Phi_j^* W) = i(\lambda - \lambda_j)\Phi_j^* W + \frac{1}{2}\left(WH(u\Phi_j^*) + \Phi_j^* H(uW) \right) - i\lambda\Phi_j^*. \tag{4.10}$$

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Using then Eqs. (4.10), (D.1), (5.15), (2.23), (2.24), and also the equation (5.7),

$$\int_{-\infty}^{\infty} \Phi_j u \, dx = 2\pi i \lambda_j \tag{4.11}$$

one obtains Eq. (4.8). (Note that Eq. (4.8) can also be obtained by taking the variational derivative with respect to λ_j of Eq. (7.17) below with $n = 2$ and then using the relation $(\delta \mathrm{Re}\gamma_j / \delta u)_x = (1/2\pi i)\delta u / \delta \lambda_j$, which follows from Eq. (4.4).)

Now, taking the inner product of Eq. (4.5) with $u(x)$ and using Eqs. (4.6)–(4.8), one obtains the expansion of the background potential over the SE to be as follows:

$$u(x) = \left[\frac{1}{2\pi i} \int_0^{\infty} \beta(\lambda) N(x, \lambda) M^*(x, \lambda) d\lambda - \sum_j \frac{\Phi_j^* \Phi_j}{2\lambda_j} \right] + \text{c.c.} \tag{4.12}$$

Using Eq. (A.6) of Appendix A, where one has to put $x = y$, and Eq. (2.10) to transform the integrand in Eq. (4.12), one can obtain an alternative expression for $u(x)$:

$$u(x) = \int_0^{\infty} (N^*(x, \lambda) N(x, \lambda) - 1) d\lambda - \sum_j \frac{\Phi_j^*(x) \Phi_j(x)}{\lambda_j}. \tag{4.13}$$

Another derivation of this equation, that uses the completeness relation (2.40) for the Jost functions, is given in Appendix B. Note that in contrast to the expansion (4.4) for $\delta u(x)$, the expansion (4.12) for $u(x)$ itself involves only quadratic combinations of the Jost functions, but not the terms $(\delta \mathrm{Re}\gamma_j / \delta u(x))$. The latter terms are the counterparts of the so called *derivative* (or associated) states (cf. Eq. (3.13)) that are required for completeness of the set of the SE [13]. This situation, when the derivative states are absent in the expansion of the potential, is typical for other $1 + 1$ integrable equations as well (see, e.g., [17] and references therein).

Finally, we mention another useful application of the SE. Notice that the linearized BO

$$q_t + 2(u_0 q)_x + H q_{xx} = 0 \tag{4.14}$$

is satisfied by the following functions, related to the SE in a simple way

$$\psi(x, t, \lambda) = e^{i\lambda^2 t}(N\overline{N}^*)_x \qquad F_j = \frac{1}{\lambda_j}(\Phi_j^* \Phi_j)_x \qquad G_j = \left(\frac{\delta \mathrm{Re}\gamma_j}{\delta u} \right)_x. \tag{4.15}$$

This can be shown straightforwardly by using the equations of the Lax pair for N, $\overline{N}$, and Φ_j [5]. Similarly, one can show that the quantities

$$\bar{\psi} = e^{i\lambda^2 t} N M^* \qquad \bar{F}_j = \frac{1}{\lambda_j} \Phi_j^* \Phi_j \qquad \bar{G}_j = \frac{\delta \mathrm{Re}\gamma_j}{\delta u} \tag{4.16}$$

satisfy an equation that is adjoint to the linearized BO:

$$\bar{q}_t + 2u_0 \bar{q}_x + H \bar{q}_{xx} = 0. \tag{4.17}$$

Then completeness of the set of the SE implies that the initial value problem, given by Eq. (4.14) and an initial condition

$$q(x, 0) = q_0(x) \tag{4.18}$$

has the following solution:

$$q(x, t) = \int_0^{\infty} d\lambda \left(\psi(x, t, \lambda) \hat{q}^*(\lambda) + \text{c.c.} \right) + \sum_j \left(G_j(x, t) \hat{q}_j^{(G)} - F_j(x, t) \hat{q}_j^{(F)} \right) \tag{4.19}$$

where

$$\hat{q}(\lambda) = -\frac{1}{2\pi i \lambda} \int_{-\infty}^{\infty} q_0(x) \tilde{v}(x, 0, \lambda) dx \qquad (4.20a)$$

$$\hat{q}_j^{(G)} = \int_{-\infty}^{\infty} q_0(x) \tilde{F}_j(x, 0) dx \qquad (4.20b)$$

$$\hat{q}_j^{(F)} = \int_{-\infty}^{\infty} q_0(x) \tilde{G}_j(x, 0) dx. \qquad (4.20c)$$

The SE can also be used to construct the solution of an inhomogeneous version of Eq. (4.14), which arises when one considers a perturbed BO equation. This problem is addressed in Section 6 below.

5 Orthogonality of the SE

In this Section, we will give a direct proof of the orthogonality relations among the SE. For the continuous spectrum, such a relation is suggested by the formal indentity $\delta\beta(\lambda)/\delta\beta(\lambda') = \delta(\lambda - \lambda')$, which upon using Eqs. (3.3) and (4.4) becomes:

$$\int_{-\infty}^{\infty} N^*(x, \lambda) M(x, \lambda) \frac{\partial}{\partial x} [N(x, \lambda') \overline{N}^*(x, \lambda')] dx = 2\pi i \lambda \delta(\lambda - \lambda'). \qquad (5.1)$$

Similarly, the identities $\partial\lambda_j/\partial\lambda_k = \partial \mathrm{Re}\gamma_j/\partial \mathrm{Re}\gamma_k = \delta_{jk}$ both imply

$$\int_{-\infty}^{\infty} \Phi_j^*(x) \Phi_j(x) \frac{\partial}{\partial x} \left(\frac{\delta Re\gamma_k}{\delta u(x)} \right) dx = \lambda_j \delta_{kj}. \qquad (5.2)$$

Below we present a direct proof of Eq. (5.1), while transferring the specific details of the proof of Eq. (5.2) to Appendix D. The idea of the proof given here is similar to the corresponding proof for the KP-II equation [18] (see Section 8 for a detailed comparison between these two proofs). Thus, giving such a proof for the BO is intended to stress the similarity of this latter equation with $2 + 1$ integrable equations [5].

We start by putting the l.h.s. of Eq. (5.1) in the following form:

$$\text{l.h.s. of } (5.1) = \frac{1}{2} \int_{-\infty}^{\infty} dx (A\tilde{B} + \tilde{A}B) + \frac{1}{2} A\tilde{A}|_{-\infty}^{\infty} \qquad (5.3)$$

where

$$A = M(x, \lambda) \overline{N}^*(x, \lambda') \qquad B = M(x, \lambda) \overline{N}_x^*(x, \lambda') - M_x(x, \lambda) \overline{N}^*(x, \lambda') \qquad (5.4a)$$

$$\tilde{A} = N^*(x, \lambda) N(x, \lambda') \qquad \tilde{B} = N^*(x, \lambda) N_x(x, \lambda') - N_x^*(x, \lambda) N(x, \lambda'). \qquad (5.4b)$$

From Eqs. (2.4), (2.5), and their analogue for M, one finds the following relations:

$$A_x = i(\lambda - \lambda')A + \frac{1}{2} \left(M H(u\overline{N}^*) + \overline{N}^* H(uM) \right) - i\lambda \overline{N}^* + i\lambda' M \qquad (5.5)$$

$$B = -i(\lambda + \lambda')A + \frac{1}{2} \left(M H(u\overline{N}^*) - \overline{N}^* H(uM) \right) - iuA + i\lambda' M + i\lambda \overline{N}^*. \qquad (5.6)$$

Let us now compute HA_x. To this end, we first note that

$$P^+ \left(\overline{N}^* H(uM) \right) = iP^+ \left(uM\overline{N}^* \right) \qquad (5.7)$$

where we have used the relations

$$H\overline{N}^* = -i\overline{N}^* + i \qquad P^+\overline{N}^* = \frac{1}{2}. \qquad (5.8)$$

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Similarly.

$$P^- \left(MH(u\overline{N}^*) \right) = -iP^- \left(u\overline{N}^* M \right).$$ (5.9)

Using Eqs. (5.7)–(5.9) and the identities $P^\pm = \frac{1}{2} \pm \frac{1}{2i} H$, we find:

$$iHA_x = -(\lambda - \lambda')HA - \frac{1}{2} \left(MH(u\overline{N}^*) - \overline{N}^* H(uM) \right) - iuA - i\lambda \overline{N}^* - i\lambda' M - i(\lambda - \lambda').$$ (5.10)

Adding Eqs. (5.6) and (5.10), we obtain the key relation:

$$iHA_x - B = -(\lambda - \lambda')HA - i(\lambda + \lambda')A - 2iuA - i(\lambda - \lambda').$$ (5.11)

Similarly to Eq. (5.11), one obtains

$$iH\tilde{A}_x - \tilde{B} = (\lambda - \lambda')H\tilde{A} - i(\lambda + \lambda')\tilde{A} - 2iu\tilde{A}.$$ (5.12)

Let us remark that the important difference between Eqs. (5.11) and (5.12) is in the last term on the r.h.s. of (5.11). To make the reason for its occurrence in (5.11) clear, first recall that $A = M\overline{N}^*$ and then note that the large-x asymptotics of M and $\overline{N}$ contain constant terms, and hence so do $P^\pm \overline{N}$ etc. (cf. Eq. (5.8)). On the contrary, $\tilde{A} = N^* N$, and $P^\pm N$ do not contain constant terms (cf. Eq. (3.2)), hence the term $i(\lambda + \lambda')$ is absent in (5.12).

Using Eqs. (5.11) and (5.12), one can cast the first term on the r.h.s. of (5.3) into the following form:

$$\int_{-\infty}^{\infty} dx(A\tilde{B} + \tilde{A}B) =$$ (5.13)

$$i(\lambda + \lambda') \int_{-\infty}^{\infty} dx\tilde{A} + i \int_{-\infty}^{\infty} dx(AH\tilde{A}_x - \tilde{A}HA_x) - (\lambda - \lambda') \int_{-\infty}^{\infty} dx(\tilde{A}HA + AH\tilde{A}).$$

Consider first the second term on the r.h.s. of (5.13) and note the following identities:

$$Hg_x = (Hg)_x$$ (5.14)

$$\int_{-\infty}^{\infty} dx(gHh + hHg) = g_0 \int_{-\infty}^{\infty} dxHh + h_0 \int_{-\infty}^{\infty} dxHg$$ (5.15)

where functions $g(x)$ and $h(x)$ are arbitrary except for the condition that for each of them, the asymptotics at plus and minus infinities coincide:

$$g_0 = \lim_{x \to \pm\infty} g(x) \qquad h_0 = \lim_{x \to \pm\infty} h(x).$$ (5.16)

In (5.16), the limits are taken in the sense of distributions, i.e. terms like $e^{i\lambda x}$ are considered to be zero as $|x| \to \infty$. Using then Eqs. (5.14), (5.15) and the asymptotics of N, M, and $\overline{N}$, we obtain:

$$\int_{-\infty}^{\infty} dx(AH\tilde{A}_x - \tilde{A}HA_x) = AH\tilde{A}\big|_{-\infty}^{\infty} = 0$$ (5.17)

where, again, we have neglected the terms $e^{i\lambda x}$ etc. for $x \to \pm\infty$ (see, however, the discussion at the end of this Section). Next, using (5.15) and the asymptotics of A and $\tilde{A}$, we transform the last term in (5.13) as follows:

$$(\lambda - \lambda') \int_{-\infty}^{\infty} dx(\tilde{A}HA + AH\tilde{A}) = (\lambda - \lambda') \int_{-\infty}^{\infty} H\tilde{A}dx.$$ (5.18)

To simplify it, let us integrate the analogue of Eq. (5.10) for $\tilde{A}$, whence one obtains that the r.h.s. of (5.18) vanishes. In arriving at this result, one needs to also use Eqs. (5.14), (5.15) and the identity $HN = iN$.

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Now, the value of the first term on the r.h.s. of Eq. (5.13) follows directly from Eq. (2.37). Thus we find that

$$\int_{-\infty}^{\infty} dx(A\tilde{B} - \tilde{A}B) = 4\pi i\lambda\delta(\lambda - \lambda').$$

(5.19)

Finally, using the asymptotics of M, N, and $\overline{N}$ for $x \to \pm\infty$, one can see that the contribution from the boundaries on the r.h.s. of Eq. (5.3) vanishes in the sense of distributions. Thus from Eqs. (5.3) and (5.19), Eq. (5.1) follows.

Using the above method, one can also prove orthogonality relations between other quadratic combinations of the Jost functions. Since some of these relations are used in the calculation of the Poisson brackets in Section 7, we will list them below:

$$\int_{-\infty}^{\infty} dx\,\overline{N}N^*(\lambda)\frac{\partial}{\partial x}\left[\overline{N}^*N(\lambda')\right] = 2\pi i\lambda\delta(\lambda - \lambda') - \beta(\lambda)\beta^*(\lambda')\Theta(\lambda - \lambda')$$

(5.20)

$$\int_{-\infty}^{\infty} dx\,M^*N(\lambda)\frac{\partial}{\partial x}\left[\overline{N}^*N(\lambda')\right] = 0$$

(5.21)

$$\int_{-\infty}^{\infty} dx\,N^*N(\lambda)\frac{\partial}{\partial x}[\overline{N}^*N(\lambda')] = \beta^*(\lambda')\Theta(\lambda - \lambda')$$

(5.22)

$$\int_{-\infty}^{\infty} dx\,N^*N(\lambda)\frac{\partial}{\partial x}\left[N^*N(\lambda')\right] = 0$$

(5.23)

$$\int_{-\infty}^{\infty} dx\,MN^*(\lambda)\frac{\partial}{\partial x}\left(\frac{\delta\text{Re}\gamma_j}{\delta u(x)}\right) = \int_{-\infty}^{\infty} dx\,\overline{N}N^*(\lambda)\frac{\partial}{\partial x}\left(\frac{\delta\text{Re}\gamma_j}{\delta u(x)}\right) = 0$$

(5.24)

$$\int_{-\infty}^{\infty} dx\,MN^*(\lambda)\frac{\partial}{\partial x}(\Phi_j\Phi_j^*) = \int_{-\infty}^{\infty} dx\,\overline{N}N^*(\lambda)\frac{\partial}{\partial x}(\Phi_j\Phi_j^*) = 0.$$

(5.25)

Here $\Theta(\lambda - \lambda')$ is the Heaviside step function: $\Theta(\lambda - \lambda') = 1$ for $\lambda > \lambda'$, and $\Theta(\lambda - \lambda') = 0$ for $\lambda < \lambda'$. Note also that Eq. (5.21) is equivalent to the formal identity $\delta\beta^*(\lambda)/\delta\beta(\lambda') = 0$, and the first equations in (5.24) and (5.25) are equivalent to $\delta\beta(\lambda)/\delta\lambda_j = 0$, etc. (cf. Eqs. (3.3) and (4.4)). Eqs. (5.25) are obtained straightforwardly, while Eqs. (5.24) can be derived following the lines of Appendix D.

Before concluding this Section, let us remark on a subtle point in the derivation of Eqs. (5.20)–(5.23), which we could avoid when deriving Eq. (5.1). Indeed, in the latter case, the integral in the middle part in Eq. (5.17) vanished in the sense of distributions. However, this would not be the case if one considers, e.g., the derivation of Eq. (5.20). The reason for this lies in the large-x asymptotics of the quantities A and $\tilde{A}$, which for Eq. (5.20) are defined differently than for Eq. (5.1). Indeed, in the case of Eq. (5.20), one should define these quantities to be

$$A = \overline{N}(\lambda)\overline{N}^*(\lambda') \qquad \tilde{A} = N^*(\lambda)N(\lambda').$$

(5.26)

Their asymptotics are

$$A \to 1 \qquad \tilde{A} \to e^{i(\lambda - \lambda')x} \qquad (x \to +\infty)$$

(5.27a)

$$A \to \left(1 - \frac{\beta(\lambda)}{\Gamma(\lambda)}e^{i\lambda x}\right)\left(1 - \frac{\beta^*(\lambda')}{\Gamma^*(\lambda')}e^{-i\lambda'x}\right) \qquad \tilde{A} \to \frac{e^{i(\lambda' - \lambda)x}}{\Gamma^*(\lambda)\Gamma(\lambda')} \qquad (x \to -\infty).$$

(5.27b)

One can rewrite the term $H\tilde{A}$ as follows:

$$H\tilde{A} = \frac{1}{\pi}\left(\int_{-\infty}^{x-L} + \int_{x+L}^{\infty} \frac{dy\,\tilde{A}(y)}{y - x}\right) + \frac{1}{\pi}P\int_{-L}^{L}\frac{\tilde{A}(y + x)dy}{y}$$

(5.28)

where $1 \ll L \ll |x|$ (we consider the width of the potential $u(x)$ to be $O(1)$). Let us take the limit in Eq. (5.28) as $x \to -\infty$. Then, using Eq. (5.27b), one can see that the first two integrals in Eq. (5.28)

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give a contribution $O(x^{-1})$, whereas in the last integral, one can replace $\tilde{A}(y-x)$ by its asymptotic value. This yields:

$$H\tilde{A} = \frac{ie^{i(\lambda'-\lambda)x}}{\Gamma^*(\lambda)\Gamma(\lambda')}\,\mathrm{sgn}(\lambda'-\lambda) + O\left(\frac{1}{x}\right) \qquad (x \to -\infty)\,. \tag{5.29}$$

Using Eq. (5.29) and its analogue for $x \to +\infty$, as well as the asymptotics (5.27), one obtains that for A and $\tilde{A}$ as in (5.26), one has

$$\int_{-\infty}^{\infty} dx(AH\tilde{A}_x - \tilde{A}HA_x) = -i3(\lambda)3^*(\lambda')\,\mathrm{sgn}(\lambda'-\lambda)\,. \tag{5.30}$$

Note that this is nonzero, in contrast to Eq. (5.17), which gave zero instead. Now, continuing, we combine Eqs. (5.30), (5.13), (5.3), and (5.27) to obtain Eq. (5.20). Calculations of the counterparts of Eq. (5.17) for the other orthogonality relations are similar.

6 Perturbation Theory for the BO Equation

The most important application of the developed machinery in the context of physics is, certainly, finding the variation of the potential $\delta u(x, t > 0)$ that could be caused either by a perturbation to the BO or by a small deviation of the initial profile from a certain special solution (e.g., N-soliton solution). Let us first consider the perturbed BO in the form:

$$u_t + 2uu_x + Hu_{xx} = \epsilon R[u, x, t] \tag{6.1}$$

where ϵ is a small parameter characterizing the strength of the perturbation R. Let $u = u_0 + \epsilon\delta u + \cdots$, where $u_0(x)$ is the background solution in the absence of a perturbation. Then $\delta u(x)$ is given by Eq. (4.4), where the Jost functions should be computed using the unperturbed "potential" $u_0(x)$, and the variations of the scattering data can be found as follows. First, one has

$$\delta\beta_t(\lambda, t) = \int_{-\infty}^{\infty} dx\frac{\delta\beta}{\delta u}u_t = \tag{6.2}$$

$$\int_{-\infty}^{\infty} dx\frac{\delta\beta}{\delta u}[(-2uu_x - Hu_{xx}) + \epsilon R] = i\lambda^2\delta\beta + \epsilon\int_{-\infty}^{\infty} dx\frac{\delta\beta}{\delta u}R\,.$$

In arriving at the last expression in Eq. (6.2), we have used the well known time evolution equation for β in the unperturbed BO: $\beta_t = i\lambda^2\beta$ [5]. Finally, using Eq. (3.3), we obtain:

$$\delta\beta_t = i\lambda^2\delta\beta + i\epsilon\int_{-\infty}^{\infty} MN^*(x, \lambda)R[u, x, t]dx\,. \tag{6.3}$$

Similarly, one derives the following equation for $\lambda_j = \lambda_{j0} + \delta\lambda_j$ and $\mathrm{Re}\gamma_j = (\mathrm{Re}\gamma_j)_0 + \delta\mathrm{Re}\gamma_j$:

$$(\lambda_j)_t = \frac{\epsilon}{2\pi\lambda_j}\int_{-\infty}^{\infty} \Phi_j\Phi_j^* R\,dx \tag{6.4}$$

$$(\mathrm{Re}\gamma_j)_t = 2\lambda_j + \epsilon\int_{-\infty}^{\infty} \frac{\delta\mathrm{Re}\gamma_j}{\delta u}R\,dx\,. \tag{6.5}$$

However, if we now substitute $\delta\lambda_j$ and $\delta\mathrm{Re}\gamma_j$, found from Eqs. (6.4) and (6.5), into Eq. (4.4), we will, in general, find that these terms may grow (usually, linearly) with t, which will invalidate the expansion (4.4) for $t = O(1/\epsilon)$. The remedy for this situation is well known: Instead of taking the background solution $u_0(x)$ with *fixed* parameters λ_{j0} and γ_{j0}, one takes it with the parameters $\lambda_j(t) = \lambda_{j0} + \delta\lambda_j(t)$,

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$\gamma_j(t) = \gamma_{j0} + \delta\gamma_j(t)$, which evolve according to Eqs. (6.4) and (6.5). Thus one obtains the solution of Eq. (6.1) in the following form:

$$u = u_0[\lambda_j(t), \mathrm{Re}\gamma_j(t)] - \frac{1}{2\pi}\frac{\partial}{\partial x}\int_0^\infty \frac{\delta\beta\,\overline{N}^*N - \text{c.c.}}{\lambda}d\lambda \qquad (6.6)$$

where

$$\delta\beta(\lambda,t) = \delta\beta(\lambda,0)\,e^{i\lambda^2 t} - i\epsilon\int_0^t ds\,e^{i\lambda^2(t-s)}\int_{-\infty}^\infty dx\,M N^*(x,s)R[u,x,s]. \qquad (6.7)$$

The following four remarks about Eq. (6.6) are in order. First, in the case of a pure N-soliton background potential $u(x)$, this result has been obtained by the direct perturbation method in [4]. In the case of the 1-soliton background potential, an equivalent result was obtained even earlier in [2]. Second, Eq. (4.19) becomes a particular case of Eq. (6.6) when $R \equiv 0$ and when one does *not* include the potentially secular terms (see above) in the background solution. Third remark: The second term on the r.h.s. of Eq. (6.6) describes the evolution of purely radiational modes. Obviously, expansion (6.6) is valid only for such times when the integral term in (6.7) can still be considered to be small. However, even when $\delta\beta(\lambda,t)$ is small, the integral in (6.6) could still possibly produce secular growth in time, as it occurs for the KdV (the celebrated tail recoil phenomenon; see, e.g., [19]). The question of whether this also occurs for the BO is outside the scope of this paper and will be addressed elsewhere.

Fourth remark: When $u_0(x)$ is a pure N-soliton solution, then $\beta(\lambda) \equiv 0$ and the Jost functions in (6.6) can be found explicitly. One first finds Φ_j from (2.26) and then $\overline{N}$ (which for $\beta \equiv 0$ also equals M) from (2.25). To obtain $N(x,\lambda)$ in this case, one notices from (2.33) and (2.29) that

$$N(x,\lambda) = e^{i\lambda x}\overline{N}(x,0). \qquad (6.8)$$

Thus, all the quantities in (6.6) can be explicitly computed. Below we will list them for the case of the 1-soliton potential

$$u_0(x,t) = \frac{2/a}{(x - x_0 - at)^2 + 1/a^2} \qquad (6.9)$$

where (see [7])

$$\lambda_1 = -\frac{a}{2} \qquad \gamma_1 = -x_0 + \frac{i}{a}. \qquad (6.10)$$

Then from Eqs. (2.25), (2.26), and (6.8), one has:

$$\Phi_1(x) = \frac{1}{x + \gamma_1} \qquad (6.11)$$

$$\overline{N}(x,\lambda) = M(x,\lambda) = 1 - \frac{i\Phi_1}{\lambda - \lambda_1} \qquad (6.12)$$

$$N(x,\lambda) = e^{i\lambda x}\left(1 + \frac{i}{\lambda_1}\Phi_1\right). \qquad (6.13)$$

7 Hamiltonian Structure of the BO Equation

In this section, we show that the BO equation is an infinite–dimensional, completely integrable Hamiltonian system, and present a set of its canonical action–angle variables. In Ref. [8], we performed the corresponding calculations using the antisymmetrized version of the Gardner Poisson bracket (PB). However, as it was first pointed out in [20], both the Gardner bracket and its antisymmetrized version do not satisfy the Jacobi identity, and hence neither of them constitutes a proper form of the PB for

the class of long-wave equations, to which the BO equation belongs. Therefore, below we will use the so called Arkad'ev–Pogrebkov–Polivanov (APP) form of the PB:

$$\{F, G\} = \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{\delta F}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta G}{\delta u(x)} - \frac{\delta G}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta F}{\delta u(x)} \right] dx \qquad (7.1)$$

$$- \frac{1}{2} \left(\frac{\delta F}{\delta u(-)} \frac{\delta G}{\delta u(-)} - \frac{\delta F}{\delta u(-)} \frac{\delta G}{\delta u(-)} \right)$$

where $F \equiv F[u]$ and $G \equiv G[u]$ are functionals of u, and $\delta F / \delta u(\pm) \equiv \lim_{x \to \pm\infty} \delta F / \delta u(x)$. The APP bracket (7.1) was first introduced in [21], where it was pointed out that it satisfies the Jacobi identity and, moreover, separates the canonical variables corresponding to the discrete and continuous parts of the spectrum. Another well-known form of the PB that also satisfies the Jacobi identity is the Faddeev–Takhtajan (FT) bracket [20]; it has a plus, instead of the minus, in front of the boundary terms in (7.1). However, the FT bracket does not separate the discrete and continuous action–angle variables. A detailed comparison of the APP and FT brackets can be found in Refs. [15, 22]. Here we only make two remarks: First, all of the aforementioned versions of the PB's lead to the same evolution equations for the field u and the action–angle variables [15]. For instance, the BO equation (1.1) can be rewritten as

$$u_t = \{u, \mathcal{H}\} \qquad (7.2)$$

where the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = - \int_{-\infty}^{\infty} \left(\frac{u^3}{3} + \frac{1}{2} u H u_x \right) dx \qquad (7.3)$$

and $\{\ldots\}$ denote any of the PB's mentioned above. Second, using the APP bracket rather than the FT one is more convenient technically, in the sense that the former does not require taking account of the terms like $\lambda \delta(\lambda)$ in the canonical brackets (Eqs. (7.12)–(7.16) below), whereas the latter does require that (cf. [15, 20]). Thus, since below we are using the APP bracket, we did not need to keep terms like $\lambda \delta(\lambda)$ in the orthogonality relations (5.1) and (5.20)–(5.25) in Section 5.

Following Ref. [8], we define the action–angle variables corresponding to the continuous spectrum to be the following:

$$p(\lambda) = \frac{\beta^*(\lambda)\beta(\lambda)}{2\pi\lambda} = i \left(\ln \Gamma(\lambda) \right)_\lambda \qquad (7.4)$$

$$q(\lambda) = \frac{1}{2i} \ln \frac{\beta(\lambda)}{\beta^*(\lambda)} + \frac{i}{2} \ln \Gamma(\lambda) \qquad (7.5)$$

and for the discrete spectrum, we define them to be

$$p_j = \lambda_j \qquad (7.6)$$

$$q_j = 2\pi \text{Re } \gamma_j . \qquad (7.7)$$

In order to calculate the PB's for these action-angle variables, we first note from Eqs. (2.16), (3.3), and (A.6) that the variational derivatives of $p(\lambda)$ and $q(\lambda)$ are given by

$$\frac{\delta p(\lambda)}{\delta u(x)} = -\frac{1}{2\pi i \lambda} (\beta^* N^* M - \beta N M^*) = -\frac{1}{2\pi i \lambda} (\beta^* N^* \overline{N} - \beta N \overline{N}^*) \qquad (7.8)$$

$$\frac{\delta q(\lambda)}{\delta u(x)} = \frac{1}{2} \left(\frac{N^* M}{\beta} + \frac{N M^*}{\beta^*} - N^* N \right) = \frac{1}{2} \left(\frac{N^* \overline{N}}{\beta} + \frac{N \overline{N}^*}{\beta^*} + N^* N \right) \qquad (7.9)$$

where in passing to the last expression in Eq. (7.9). the relation (2.10) has been used. Using the asymptotics (2.2). (2.3). (2.13). and (2.14). one finds from Eqs. (7.8) and (7.9) that

$$\frac{\delta p(\lambda)}{\delta u(\mp)} = 0 \qquad \frac{\delta q(\lambda)}{\delta u(\mp)} = \mp\frac{1}{2}. \tag{7.10}$$

A similar caluculation. using Eqs. (2.22). (3.5). and (3.15). gives the corresponding boundary values for the discrete variables:

$$\frac{\delta p_j}{\delta u(\mp)} = 0 \qquad \frac{\delta q_j}{\delta u(\mp)} = 0. \tag{7.11}$$

It is now straightforward to calculate the various PB's between $p(\lambda)$ and $q(\lambda)$. Indeed. substituting Eqs. (7.8)–(7.10) into Eq. (7.1) and using the orthogonality relations (5.1) and (5.20)–(5.25). we obtain:

$$\{p(\lambda). q(\mu)\} = -\delta(\lambda - \mu) \tag{7.12}$$

$$\{p(\lambda), p(\mu)\} = \{q(\lambda). q(\mu)\} = 0. \tag{7.13}$$

In the same way, we can calculate the PB's for the discrete variables. as well as those between the discrete and continuous variables. The result is as follows:

$$\{p_j, q_k\} = \delta_{jk} \tag{7.14}$$

$$\{p_j, p_k\} = \{q_j. q_k\} = 0 \tag{7.15}$$

$$\{p_j, q(\lambda)\} = \{p_j, p(\lambda)\} = \{q_j, q(\lambda)\} = \{q_j, p(\lambda)\} = 0. \tag{7.16}$$

In terms of the action-angle variables. the nth conserved quantity of the BO equation [7] can be represented by

$$I_n = (-1)^n \int_0^\infty \lambda^{n-1} p(\lambda) d\lambda + 2\pi \sum_j (-p_j)^{n-1} \qquad (n = 1, 2, ...). \tag{7.17}$$

Since by Eqs. (7.13) and (7.16), $p(\lambda)$ and p_j commute with each other, then the PB between any I_m and I_n vanishes:

$$\{I_m, I_n\} = 0 \qquad (m, n = 1, 2, ...). \tag{7.18}$$

The existence of an infinite number of commuting integrals of motion is an important feature of a completely integrable Hamiltonian system. We note that Eq. (7.18) was derived in earlier works [23] and [24] by different means.

Now, in view of the relation $\mathcal{H} = -I_3$, the Hamiltonian (7.3) takes the following simple form:

$$\mathcal{H} = \int_0^\infty \lambda^2 p(\lambda) d\lambda - 2\pi \sum_j p_j^2. \tag{7.19}$$

Then, using Eqs. (7.12)–(7.16) and (7.19), one finds the time evolution of the action-angle variables to be as follows:

$$\frac{\partial p(\lambda)}{\partial t} = \{p(\lambda), \mathcal{H}\} = 0 \qquad \frac{\partial q(\lambda)}{\partial t} = \{q(\lambda), \mathcal{H}\} = \lambda^2 \tag{7.20}$$

$$\frac{\partial p_j}{\partial t} = \{p_j, \mathcal{H}\} = 0 \qquad \frac{\partial q_j}{\partial t} = \{q_j, \mathcal{H}\} = 4\pi p_j. \tag{7.21}$$

The above system of equations is integrated trivially and it reproduces the time evolution of the scattering data derived by the IST [5]. This proves the complete integrability of the BO equation as an infinite-dimensional Hamiltonian dynamical system.

It will be appropriate here to remark on some useful properties of the conserved quantities associated with the SE, introduced in Section 4. It follows from Eqs. (2.20), (2.29), (7.8), (7.17) and (A.6) that

the variational derivative of the nth conserved quantity of the BO equation is expressed in terms of the SE as follows:

$$\frac{\delta I_n}{\delta u(x,t)} = (n-1)\left[(-1)^n \int_0^\infty \lambda^{n-2}\left(N^*(x,t,\lambda)N(x,t,\lambda) - 1\right)d\lambda\right.$$

$$\left. + \sum_j (-\lambda_j)^{n-3}\Phi_j^*(x,t)\Phi_j(x,t)\right] \qquad (n \geq 2).$$

(7.22)

Remarkably, this quantity is found to give an explicit solution[1] to the adjoint of the linearized BO. Eq. (4.17), since both N^*N and $\Phi_j^*\Phi_j$ satisfy that equation. In particular, if we take $n = 2$ in Eq. (7.22) and use the relation $\delta I_2/\delta u = u$, we can also obtain the expansion (4.13).

The following relation, which is derived with the use of Eq. (2.4), is also worth noting:

$$\int_{-\infty}^\infty \left(N^*(x,t,\lambda)N(x,t,\lambda) - 1\right)dx = -\frac{J^*(\lambda)J(\lambda)}{2\pi\lambda}.$$

(7.23)

Integrating Eq. (7.22) with respect to x and substituting Eqs. (7.23) and (2.39) into the resultant expression, one has

$$\int_{-\infty}^\infty \frac{\delta I_n}{\delta u(x,t)}dx = (n-1)I_{n-1} \qquad (n \geq 2)$$

(7.24)

where we have used the expression of I_{n-1} in terms of the scattering data, as found from Eqs. (7.4), (7.6) and (7.17). The above relation shows that $\delta I_n/\delta u$ is proportional to the conserved density of I_{n-1}. While Eq. (7.24) has been obtained earlier in [25] by analyzing the recurrence formula for the conserved quantities, the derivation presented here gives an independent proof relying on the properties of the SE.

As is well known (see, e.g., [15]), the accounting for the boundary terms in the PB, as in Eq. (7.1), affects the value of any PB containing the first conserved quantity, which for the BO equation is

$$I_1 = \int_{-\infty}^\infty u\,dx = -\int_0^\infty p(\lambda)d\lambda + 2\pi n$$

(7.25)

where n is the total number of the bound states. Obviously, one has $\delta I_1/\delta u \equiv 1$. Then for any functional F with nonvanishing boundary values, Eq. (7.1), yields:

$$\{F, I_1\} = -\left(\frac{\delta F}{\delta u(+)} - \frac{\delta F}{\delta u(-)}\right).$$

(7.26)

Among the action-angle variables (7.4)–(7.7), only the continuous angle variable $q(\lambda)$ has nonvanishing boundary values, as seen from Eqs. (7.10) and (7.11). Hence, we put $F = q(\lambda)$ in the above expression and see that

$$\{q(\lambda), I_1\} = -1$$

(7.27)

i.e. I_1 does not annihilate the PB (7.1). In contrast to this, one can easily show that the FT bracket between I_1 and $q(\lambda)$ or any other allowed functional [20, 26] vanishes, and hence the FT bracket has an annihilator (as is also the case with the FT bracket for the KdV equation).

To conclude this section, we derive yet another useful relation. First, using Eqs. (A.6) and (7.8), one obtains:

$$\frac{\delta p(\lambda)}{\delta u(x)} = -\frac{\partial}{\partial \lambda}\left(N^*(x,\lambda)N(x,\lambda)\right).$$

(7.28)

Next, integrating Eq. (7.28) with respect to λ with the boundary conditions (2.20) and (2.29), one obtains

$$\int_0^\infty \frac{\delta p(\lambda)}{\delta u(x)}d\lambda = -1.$$

(7.29)

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[1]This fact was originally noticed by Case [24].

Finally, taking the variational derivative with respect to $u(x)$ of Eq. (7.25) and then substituting Eq. (7.29) and the identity $\delta I_1/\delta u \equiv 1$ into the resultant expression, we obtain the following important relation:

$$\frac{\delta n}{\delta u(x)} = 0 . \tag{7.30}$$

Let us emphasize that both Eqs. (7.29) and (7.30) are derived *only* for the case of a generic potential u (compare Eq. (2.29), which was used in the derivation of (7.29), with Eq. (2.31) for a nongeneric potential). Thus, Eq. (7.30) indicates that the number of bound states, n, corresponding to a generic background potential, is invariant under small change of the potential.

8 Conclusions and Discussion

In this work, we have developed the perturbation theory for the BO equation, (1.1), when the background potential decays at infinity. Our perturbation theory is based on the IST, which was first developed in Ref. [5]. We have also made use of the essential relation between the scattering coefficients $\beta(\lambda)$ and $f(\lambda)$, Eq. (2.35), that was found in [7], as well as the asymptotics for the Jost functions and the scattering coefficients at $\lambda \to 0$ [5, 6, 7]. The most essential results of this work[2] are as follows:

1. The variational derivatives $\delta\beta/\delta u$, $\delta\lambda_j/\delta u$, and $\delta\gamma_j/\delta u$ are given by Eqs. (3.3), (3.5), and (3.15), respectively.

2. The expansion of δu over the SE, with the variation of the scattering data being the coefficients of that expansion, is given by Eq. (4.4).

3. The completeness relation for the SE is given by Eq. (4.5); note also an independent proof of this relation in Appendix C. Various applications of Eq. (4.5) are discussed in Section 4. Among those, we note the solution of the linearized BO; see Eq. (4.19).

4. The first-order solution of the perturbed BO is given by Eq. (6.6), for the perturbation of a general form. We also give the explicit form of that solution, Eq. (6.6), for the case when the background solution is a single soliton.

5. A proof of the complete integrability of the BO as a Hamiltonian system, by using the Arkad'ev–Pogrebkov–Polivanov form of the Poisson bracket is given in Section 7.

It has been long known [5] that the IST for the BO has common features with both $1+1$ and $2+1$ integrable equations. In the remainder of this section, we will discuss these features in some detail. We will limit ourselves to pointing out only two features for each of these two classes of equations.

First, we note that the expression (3.15) for the variation of the normalization contant, $\delta\mathrm{Re}\,\gamma_j/\delta u$, involves the so-called derivative state [13], i.e. the λ-derivative of a quadratic combination of the Jost functions at the pole $\lambda = \lambda_j$; cf. Eq. (D.1). This situation is a common place for other $1+1$ equations; see, e.g., [17] and references therein.

Second, the derivation of Eq. (4.4) for $\delta u(x)$ has a number of similarities with the analogous derivations for other $1+1$ equations. The scheme of the latter derivations is as follows (see, e.g., [27, 17, 28] and references therein). First, one uses the connection formula between the Jost functions that have simple asymptotics at the opposite ends of the x-axis (such as Eqs. (4.1) and (4.5) in [12]), as well as the analyticity properties of these functions in the λ-plane, to cast the direct scattering problem for the corresponding Lax operator in the form of a Riemann-Hilbert problem. The analogue of the connection formula for the BO is Eq. (2.10), and the function that has analytic properties in the λ-plane is $W(x, \lambda)$. Next, solvability of the Riemann-Hilbert problem in the case of $1+1$ equations essentially

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[2]Most of these results have been announced in [8].

depends on whether a certain symmetry exists between the Jost functions [26] (see also a discussion in [12]). The analogue of the symmetry relation for the BO is Eq. (2.33). Finally, one considers a variation of the above Riemann-Hilbert problem and then uses the asymptotics of the Jost functions at the essential singularities in the complex λ-plane (usually, at $\lambda \to \infty$), from which asymptotics, one can determine the potential and/or its variation, δu. The corresponding asymptotics for the BO are Eqs. (2.19) and (2.20). Thus, one can see that the "ingredients" in this scheme and in the derivation of the δu in Section 4 are the same, although the order in which they are used is different. For example, in Section 4, we first used the analogue of the symmetry relation, Eq. (2.33), and only after that, used the connection formula (2.10) and the analyticity of $W(x, \lambda)$.

The most prominent similarity of the perturbation scheme for the BO with that of the $2-1$ integrable equations is the following. In order to find a perturbed Jost function, as in Eq. (2.41), that corresponds to a certain value, λ_0, of the spectral parameter, it is necessary to know the *Green's function* of the Lax operator (cf. Eq. (2.42)). That Green's function involves the unperturbed Jost functions for *all* values of λ. In contrast, to find the perturbed Jost functions in the case of the $1-1$ equations, it is always sufficient to use the method of variation of parameters, which only involves the unperturbed Jost functions for the same λ_0 (see, e.g., [14, 19]).

Another interesting similarity with the $2+1$ equations concerns the proof of the orthogonality of the SE, given in Section 5. In contrast to the $1+1$ case, we did not need to make use of the so called recursion operator, i.e. the operator whose eigenfunctions are the SE, in order to prove their orthogonality. (This appears to be consistent with the conclusion made in Ref. [29, 30], that both $2+1$ equations and the BO do not possess non-singular recursion operators.) Instead, our proof closely followed that presented in [18] for the KP-II equation (see also [31]). Indeed, a crucial role in Section 5 was played by the connection formulae (5.11) and (5.12), which are just the counterparts of the corresponding connection formula for the KP-II (see the unnumbered equation immediately preceding Eq. (65) in [18]). Note that the quadratic combinations $A, \bar{A}, B, \bar{B}$ of the Jost functions pertain to *different* values of the spectral parameter (cf. Eqs. (5.4))), similarly to the quantities defined in Eq. (65) in [18].

Acknowledgements

Appendix A.   Completeness Relation for the Jost Functions

Here we present an algebraic proof of the completeness relation (2.40) for the Jost functions without recourse to the procedure of Ref. [8] that used contour integration. The idea is analogous to that originated in the proof of the completeness relation [4] for the eigenfunctions corresponding to the N-soliton potential.

We begin with an obvious identity

$$\sum_j \frac{\Phi_j(x)}{\lambda - \lambda_j} \sum_k \frac{\Phi_k^*(y)}{\lambda - \lambda_k} = \sum_j \frac{\Phi_j(x)\Phi_j^*(y)}{(\lambda - \lambda_j)^2} \tag{A.1}$$

$$- \sum_{j \neq k} \frac{1}{\lambda_j - \lambda_k}\left(\frac{1}{\lambda - \lambda_k} - \frac{1}{\lambda - \lambda_j}\right)\Phi_j(x)\Phi_k^*(y) \,.$$

Introducing the second term on the left-hand side of Eq. (2.26) into the second term on the right-hand side of Eq. (A.1), we can transform the latter equation into the form

$$\sum_{j,k} \frac{\Phi_j(x)\Phi_j^*(y)}{(\lambda - \lambda_j)(\lambda - \lambda_k)} = i\sum_j \frac{\Phi_j(x)}{\lambda - \lambda_j} - i\sum_j \frac{\Phi_j^*(y)}{\lambda - \lambda_j} + i(x - y)\sum_j \frac{\Phi_j(x)\Phi_j^*(y)}{\lambda - \lambda_j} \tag{A.2}$$

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$$-\lambda \sum_j \frac{\Phi_j(x)\Phi_j^*(y)}{\lambda_j(\lambda - \lambda_j)^2} - \frac{1}{2\pi} \sum_j \frac{\Phi_j(x)}{\lambda - \lambda_j} \int_0^\infty \frac{\beta^*(\lambda')N^*(y,\lambda')}{\lambda' - \lambda_j} d\lambda'$$

$$-\frac{1}{2\pi} \sum_j \frac{\Phi_j^*(y)}{\lambda - \lambda_j} \int_0^\infty \frac{\beta(\lambda')N(x,\lambda')}{\lambda' - \lambda_j} d\lambda'$$

where we have used the relation $\gamma_j = \mathrm{Re}\, \gamma_j - i(1/2\lambda_j)$. It now follows from Eq. (2.25) that

$$\sum_j \frac{\Phi_j(x)}{\lambda_j} = -i\left[\overline{N}(x,0) - 1 - \frac{1}{2\pi i} \int_0^\infty \frac{\beta(\lambda)N(x,\lambda)}{\lambda - i\epsilon} d\lambda\right] \tag{A.3}$$

$$\sum_j \frac{\Phi_j(x)}{\lambda_j} \int_0^\infty \frac{\beta^*(\lambda)N^*(y,\lambda)}{\lambda - \lambda_j} d\lambda$$

$$= \int_0^\infty d\lambda \beta^*(\lambda)N^*(y,\lambda)\left[\frac{i}{\lambda}\{\overline{N}(x,\lambda) - \overline{N}(x,0)\} - \frac{1}{2\pi} \int_0^\infty \frac{\beta(\lambda')N(x,\lambda')}{\lambda'(\lambda' - \lambda - i\epsilon)} d\lambda'\right]. \tag{A.4}$$

If we put $\lambda = 0$ in Eq. (A.2) and substitute (A.3) and (A.4) into the resultant expression. we find that most terms are cancelled and we are left with the relation

$$\overline{N}(x,0)\overline{N}^*(y,0) = 1 - i(x - y)\sum_j \frac{\Phi_j(x)\Phi_j^*(y)}{\lambda_j} \tag{A.5}$$

$$-\frac{i}{2\pi} \int_0^\infty \frac{d\lambda}{\lambda}[-\beta^*(\lambda)\overline{N}(x,\lambda)N^*(y,\lambda) + \beta(\lambda)N(x,\lambda)\overline{N}^*(y,\lambda)].$$

In view of relations (2.33) and (2.35), the integrand on the right-hand side of Eq. (A.5) can be rewritten as

$$-\frac{i}{2\pi\lambda}\left(-\beta^*(\lambda)\overline{N}(x,\lambda)N^*(y,\lambda) + \beta(\lambda)N(x,\lambda)\overline{N}^*(y,\lambda)\right) \tag{A.6}$$

$$= i(x - y)N(x,\lambda)N^*(y,\lambda) - \frac{\partial}{\partial\lambda}\left(N(x,\lambda)N^*(y,\lambda)\right).$$

After integrating over λ and using the relation $\overline{N}(x,0) = N(x,0)$ (see Eqs. (2.29) and (2.31)), Eq. (A.5) becomes

$$0 = 1 - i(x - y)\sum_j \frac{\Phi_j(x)\Phi_j^*(y)}{\lambda_j} + i(x - y)\int_0^\Lambda d\lambda N(x,\lambda)N^*(y,\lambda) - N(x,\Lambda)N^*(y,\Lambda) \tag{A.7}$$

where Λ is a large positive constant, which is to be taken to be infinity at the end. If we divide both sides of Eq. (A.7) by $-i(x - y)$ and take the limit $\Lambda \to +\infty$, we arrive at Eq. (2.40) upon invoking the asymptotic (2.20) and the formula $\lim_{\Lambda \to +\infty} P \frac{e^{i\Lambda(x-y)}}{x-y} = \pi i \delta(x - y)$.

Appendix B. Alternative Derivation of Eq. (4.13)

Here we present another derivation of the expansion (4.13) for the potential that employs the completeness relation for the Jost function. We first differentiate Eq. (A.7) by y to obtain the relation

$$i\sum_j \frac{\Phi_j(x)\Phi_j^*(y)}{\lambda_j} - i(x - y)\sum_j \frac{\Phi_j(x)\frac{\partial\Phi_j^*(y)}{\partial y}}{\lambda_j} - i\int_0^\Lambda d\lambda N(x,\lambda)N^*(y,\lambda) \tag{B.1}$$

$$+i(x - y)\int_0^\Lambda d\lambda N(x,\lambda)\frac{\partial N^*(y,\lambda)}{\partial y} - N(x,\Lambda)\frac{\partial N^*(y,\Lambda)}{\partial y} = 0$$

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It follows from the equation for N (see (2.4)) that

$$\frac{\partial N^*(y,\Lambda)}{\partial y} = -i\Lambda N^*(y,\Lambda) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{u(z)N^*(z,\Lambda)}{z-y-i\epsilon}dz\,. \qquad (B.2)$$

If we substitute Eq. (B.2) into the last term on the left-hand side of Eq. (B.1) and then put $y = x$, we find that

$$i\sum_j\frac{\Phi_j(x)\Phi_j^*(x)}{\lambda_j} - i\int_0^\Lambda d\lambda\Big(N(x,\lambda)N^*(x,\lambda)-1\Big) - \frac{1}{2\pi}N(x,\lambda)\int_{-\infty}^{\infty}\frac{u(z)N^*(z,\Lambda)}{z-x-i\epsilon}dz = 0\,. \qquad (B.3)$$

Lastly, taking the limit $\Lambda \rightarrow +\infty$ and using the asymptotic (2.20) for N together with the formula $\lim_{\Lambda\rightarrow+\infty} e^{i\Lambda(x-z)}/(x-z-i\epsilon) = 2\pi i\delta(x-z)$, we arrive at (4.13).

Appendix C.   Direct Proof of the Completeness Relation for the SE

The proof of the completeness relation (4.5) almost parallels that presented in [4] for the N-soliton potential. Therefore, here we shall describe only the outline of the proof.

First, we consider the integral which stems from the continuous part of (4.5):

$$I = -i\int_0^\Lambda\frac{d\lambda}{\lambda}N(x,\lambda)\overline{N}^*(x,\lambda)N^*(y,\lambda)M(y,\lambda)\,. \qquad (C.1)$$

In what follows, we will first fix the upper limit Λ of the integration and set it to infinity in the final stage of the calculation. By virtue of (2.10), the above integral can be splitted into two parts as $I = I_1 + I_2$, where

$$I_1 = -i\int_0^\Lambda\frac{d\lambda}{\lambda}N(x,\lambda)\overline{N}^*(x,\lambda)N^*(y,\lambda)\overline{N}(y,\lambda) \qquad (C.2)$$

$$I_2 = -i\int_0^\Lambda\frac{d\lambda}{\lambda}\beta(\lambda)N(x,\lambda)\overline{N}^*(x,\lambda)N^*(y,\lambda)N(y,\lambda)\,. \qquad (C.3)$$

We modify the integral I_1 by substituting $\overline{N}$ from Eq. (2.25). Doing the multiplication under the integral term by term, one can see that the product $\overline{N}^*(x,\lambda)\overline{N}(y,\lambda)$ yields nine different terms. We introduce Eq. (A.2) into the corresponding term and then use Eq. (2.25). After adding I_2 to the resultant expression, we find that I is simplified considerably to become:

$$I = -i\int_0^\Lambda\frac{d\lambda}{\lambda}N(x,\lambda)N^*(y,\lambda)\Big[1 - i(x-y)\sum_j\frac{\Phi_j^*(x)\Phi_j(y)}{\lambda-\lambda_j} + \lambda\sum_j\frac{\Phi_j^*(x)\Phi_j(y)}{\lambda_j(\lambda-\lambda_j)^2} \qquad (C.4)$$

$$-\frac{1}{2\pi i}\int_0^\Lambda\frac{d\lambda'}{\lambda'-\lambda-i\epsilon}\Big(-\beta(\lambda')\overline{N}^*(x,\lambda')N(y,\lambda') + \beta^*(\lambda')N^*(x,\lambda')\overline{N}(y,\lambda')\Big)\Big]\,.$$

Using Eq. (A.6), the last term on the right-hand side of Eq. (C.4) can be integrated by parts. In evaluating the contribution from the upper limit of the integral, we retain the term of order λ/Λ. This gives

$$I = -i\int_0^\Lambda\frac{d\lambda}{\lambda}N(x,\lambda)N^*(y,\lambda)\Big[1 - i(x-y)\sum_j\frac{\Phi_j^*(x)\Phi_j(y)}{\lambda-\lambda_j} + \lambda\sum_j\frac{\Phi_j^*(x)\Phi_j(y)}{\lambda_j(\lambda-\lambda_j)^2} \qquad (C.5)$$

$$-N^*(x,\Lambda)N(y,\Lambda) - i(x-y)\int_0^\Lambda d\lambda N^*(x,\lambda)N(y,\lambda)$$

$$-\lambda\Big\{\frac{N^*(x,\Lambda)N(y,\Lambda)}{\Lambda} + i(x-y)\int_0^\Lambda\frac{N^*(x,\lambda')N(y,\lambda')}{\lambda'-\lambda-i\epsilon}d\lambda' + \int_0^\Lambda\frac{N^*(x,\lambda')N(y,\lambda')}{(\lambda'-\lambda-i\epsilon)^2}d\lambda'\Big\}\Big]\,.$$

Substituting Eq. (A.7) into Eq. (C.5) and adding the resultant expression to its complex conjugate, we obtain:

$$I - I^* = -i \int_0^\Lambda d\lambda N(x,\lambda) N^*(y,\lambda) \left[-i(x-y) \sum_j \frac{\Phi_j^*(x)\Phi_j(y)}{\lambda_j(\lambda - \lambda_j)} \right. \tag{C.6}$$

$$\left. + \sum_j \frac{\Phi_j^*(x)\Phi_j(y)}{\lambda_j(\lambda-\lambda_j)^2} - \frac{N^*(x,\Lambda)N(y,\Lambda)}{\Lambda} \right] + (c.c.).$$

Now, with the use of Eqs. (2.20) and (2.33), the following estimate is possible for large Λ:

$$i \frac{\partial}{\partial x} \left[\frac{N^*(x,\Lambda)N(y,\Lambda)}{\Lambda} \int_0^\Lambda d\lambda N(x,\lambda)N^*(y,\lambda) \right] + (c.c.) \tag{C.7}$$

$$= N^*(x,\Lambda)N(y,\Lambda) \int_0^\Lambda d\lambda N(x,\lambda)N^*(y,\lambda) + (c.c.) + O(\Lambda^{-1}).$$

Furthermore, it follows from Eqs. (A.7) and (2.20) that

$$N^*(x,\Lambda)N(y,\Lambda) \int_0^\Lambda d\lambda N(x,\lambda)N^*(y,\lambda) \to -\frac{i}{x-y-i\epsilon} \qquad (\Lambda \to +\infty). \tag{C.8}$$

Let the contribution from the continuous spectrum in Eq. (6.1) be J. Then Eqs. (C.6), (C.7), and (C.8) yield

$$J = \lim_{\Lambda \to +\infty} \frac{\partial}{\partial x}(I + I^*) = 2\pi\delta(x-y) \tag{C.9}$$

$$+ \frac{\partial}{\partial x} \left[\int_0^\infty d\lambda N(x,\lambda)N^*(y,\lambda) \left\{ -(x-y) \sum_j \frac{\Phi_j^*(x)\Phi_j(y)}{\lambda_j(\lambda-\lambda_j)} - i \sum_j \frac{\Phi_j^*(x)\Phi_j(y)}{\lambda_j(\lambda-\lambda_j)^2} \right\} + (c.c.) \right].$$

Lastly, upon substituting Eq. (C.9) into Eq. (4.5) and integrating the resultant expression once with respect to x, one sees that the completeness relation to be proved reduces now to the form:

$$\int_0^\infty d\lambda N(x,\lambda)N^*(y,\lambda) \left\{ (x-y) \sum_j \frac{\Phi_j^*(x)\Phi_j(y)}{\lambda_j(\lambda-\lambda_j)} + i \sum_j \frac{\Phi_j^*(x)\Phi_j(y)}{\lambda_j(\lambda-\lambda_j)^2} \right\} - (x \leftrightarrow y) \tag{C.10}$$

$$= 2\pi \sum_j \left[\frac{\Phi_j^*(y)\Phi_j(y)}{\lambda_j} \frac{\delta \text{Re } \gamma_j}{\delta u(x)} - (x \leftrightarrow y) \right]$$

where the notation $(x \leftrightarrow y)$ indicates the interchange of the variables x and y in the preceding expression.

The next step is to modify the right-hand side of Eq. (C.10) while employing the IST equations (2.25), (2.26), and (2.33) as well as the completeness relation (A.7) for the Jost functions. The starting point is the following algebraic identity

$$\sum_j \frac{\Phi_j^*(x)\Phi_j(y)}{\lambda-\lambda_j} \sum_k \frac{\Phi_k(x)\Phi_k^*(y)}{\lambda-\lambda_k} = \sum_j \frac{\Phi_j^*(x)\Phi_j(x)\Phi_j^*(y)\Phi_j(y)}{(\lambda-\lambda_j)^2} \tag{C.11}$$

$$- \sum_{j \neq k} \frac{1}{\lambda_j - \lambda_k} \left(\frac{1}{\lambda-\lambda_k} - \frac{1}{\lambda-\lambda_j} \right) \Phi_j^*(x)\Phi_j(y)\Phi_k(x)\Phi_k^*(y).$$

After manipulating the second term on the right-hand side of (C.11) following the similar procedure as that developed in [4], we find that the relation corresponding to Eq. (B7) in [4] now takes the form

$$(x-y) \sum_j \frac{\Phi_j^*(x)\Phi_j(y)}{\lambda-\lambda_j} \sum_k \frac{\Phi_k(x)\Phi_k^*(y)}{\lambda-\lambda_k} = \lambda(x-y) \sum_j \frac{\Phi_j^*(x)\Phi_j(x)\Phi_j^*(y)\Phi_j(y)}{\lambda_j(\lambda-\lambda_j)^2} \tag{C.12}$$

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$$-i\sum_{j=k}\frac{\lambda_k}{(\lambda-\lambda_k)\lambda_j(\lambda_k-\lambda_j)^2}\left[\Phi_j^*(x)\Phi_k(x)\Phi_j(y)\Phi_k^*(y)-(x\leftrightarrow y)\right]$$

$$-i\sum_{j=k}\frac{1}{\lambda-\lambda_k}\left[\Phi_k^*(x)\Phi_k(x)\left\{\frac{\Phi_k^*(y)\Phi_j(y)-\Phi_k(y)\Phi_j^*(y)}{(\lambda_j-\lambda_k)^2}-\frac{1}{2\pi}\Phi_k^*(y)\int_0^\infty\frac{\mathcal{J}(\lambda')\mathcal{N}(y,\lambda')}{(\lambda'-\lambda_k)^2}d\lambda'\right.\right.$$

$$\left.\left.-\frac{1}{2\pi}\Phi_k(y)\int_0^\infty\frac{\mathcal{J}^*(\lambda')\mathcal{N}^*(y,\lambda')}{(\lambda'-\lambda_k)^2}d\lambda'\right\}-(x\leftrightarrow y)\right]$$

$$-\sum_j\frac{1}{\lambda-\lambda_j}\left[\Phi_j^*(x)\Phi_j(y)\left\{(x-y)\sum_k\frac{\Phi_k(x)\Phi_k^*(y)}{\lambda_k}+\lambda_j(x-y)\int_0^\infty\frac{\mathcal{N}(x,\lambda')\mathcal{N}^*(y,\lambda')}{\lambda'-\lambda_j}d\lambda'\right.\right.$$

$$\left.\left.+i\lambda_j\int_0^\infty\frac{\mathcal{N}(x,\lambda')\mathcal{N}^*(y,\lambda')}{(\lambda'-\lambda_j)^2}d\lambda'\right\}-(x\leftrightarrow y)\right].$$

We differentiate Eq. (C.12) with respect to λ and then put $\lambda=0$. Substituting $\delta\mathrm{Re}\,\gamma_j/\delta u$ from Eq. (3.15) into the third term on the right-hand side of the resultant expression, we can see that most terms are cancelled and the final result leads to the relation (C.10). This completes the proof of (4.5).

Appendix D. Orthogonality of the SE corresponding to the discrete spectrum

Here we will outline the differences in the derivation of Eq. (5.2) from that of Eq. (5.1). First, use Eqs. (3.15), (2.23), and (3.13) to obtain:

$$\frac{\delta\mathrm{Re}\gamma_k}{\delta u}=\frac{1}{4\pi\lambda_k}\left\{\Phi_k\left[-\frac{1}{\lambda_k}\partial_\lambda((\lambda-\lambda_k)W^*)+\partial_\lambda^2((\lambda-\lambda_k)W^*)\right]_{\lambda=\lambda_k}+(\text{c.c.})\right\}. \tag{D.1}$$

Then, following the lines of Section 5, denote

$$A=\Phi_j W^*(\lambda)\qquad \bar{A}=\Phi_j^*\Phi_k \tag{D.2}$$

with B and $\bar{B}$ being defined accordingly (cf. Eqs. (5.4)). Note that with these notations, the sought integral is (cf. Eq. (5.3)):

$$\int_{-\infty}^\infty\Phi_j^*\Phi_j\frac{\partial}{\partial x}\left(\frac{\delta\mathrm{Re}\gamma_k}{\delta u}\right)dx=\frac{1}{8\pi\lambda_k}\left\{\left[\left(-\frac{1}{\lambda_k}\partial_\lambda+\partial_\lambda^2\right)I\right]_{\lambda=\lambda_k}+(\text{c.c.})\right\} \tag{D.3}$$

where

$$I=(\lambda-\lambda_k)\int dx(A\bar{B}+\bar{A}B) \tag{D.4}$$

Now, the analogues of Eqs. (5.11) and (5.12) take on the form

$$iHA_x+B=-(\lambda_j-\lambda)HA-i(\lambda+\lambda_j)A-2iuA \tag{D.5}$$

$$-iH\bar{A}_x+\bar{B}=-(\lambda_j-\lambda_k)H\bar{A}+i(\lambda_j+\lambda_k)\bar{A}+2iu\bar{A}. \tag{D.6}$$

From these two equations, the analogue of Eq. (5.13) becomes:

$$I=(\lambda-\lambda_k)^2\int_{-\infty}^\infty\bar{A}\,(HA-iA)\,dx. \tag{D.7}$$

Other terms in (D.7) vanish by virtue of Eq. (5.15) and the decaying behavior of A and $\bar{A}$ at $x\to\pm\infty$. Next, in the expansion of A near $\lambda=\lambda_k$, it is sufficient to take into account only the first two terms (cf.

Eq. (2.23)). Substituting the result in Eq. (D.7) and then in Eq. (D.3), and taking the limit $\lambda \to \lambda_k$, one obtains that the term in the curly brackets in Eq. (D.3) is:

$$-\frac{2}{\lambda_k} \int_{-\infty}^{\infty} dx \Phi_j^* \Phi_k \left[iH(\Phi_j \Phi_k^*) - \Phi_j \Phi_k^* \right] + \tag{D.8}$$

$$-2 \int_{-\infty}^{\infty} dx \left[\Phi_j^* \Phi_k H \left((x + \gamma_k^*) \Phi_k^* \Phi_j \right) + \Phi_j \Phi_k^* H \left((x - \gamma_k) \Phi_k \Phi_j^* \right) \right] +$$

$$+2 \int_{-\infty}^{\infty} dx \left[\frac{1}{\lambda_k} \Phi_j \Phi_k^* \Phi_j^* \Phi_k \right].$$

In deriving Eq. (D.8), we have used Eqs. (5.15) and (2.24). Using these two equations one more time, as well as the identity

$$H(xg) = \frac{1}{\pi} \int_{-\infty}^{\infty} g(y) dy + xHg \tag{D.9}$$

one transforms Eq. (D.8) into the simple form:

$$(D.8) = \frac{2}{\pi} \left| \int_{-\infty}^{\infty} \Phi_j^* \Phi_k dx \right|^2. \tag{D.10}$$

Finally, using Eq. (2.39), one obtains:

$$\text{l.h.s. of } (D.3) = \frac{1}{8\pi\lambda_k} \frac{2}{\pi} 4\pi^2 \lambda_k^2 \delta_{kj} = \lambda_k \delta_{kj}. \tag{D.11}$$

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Figure Caption

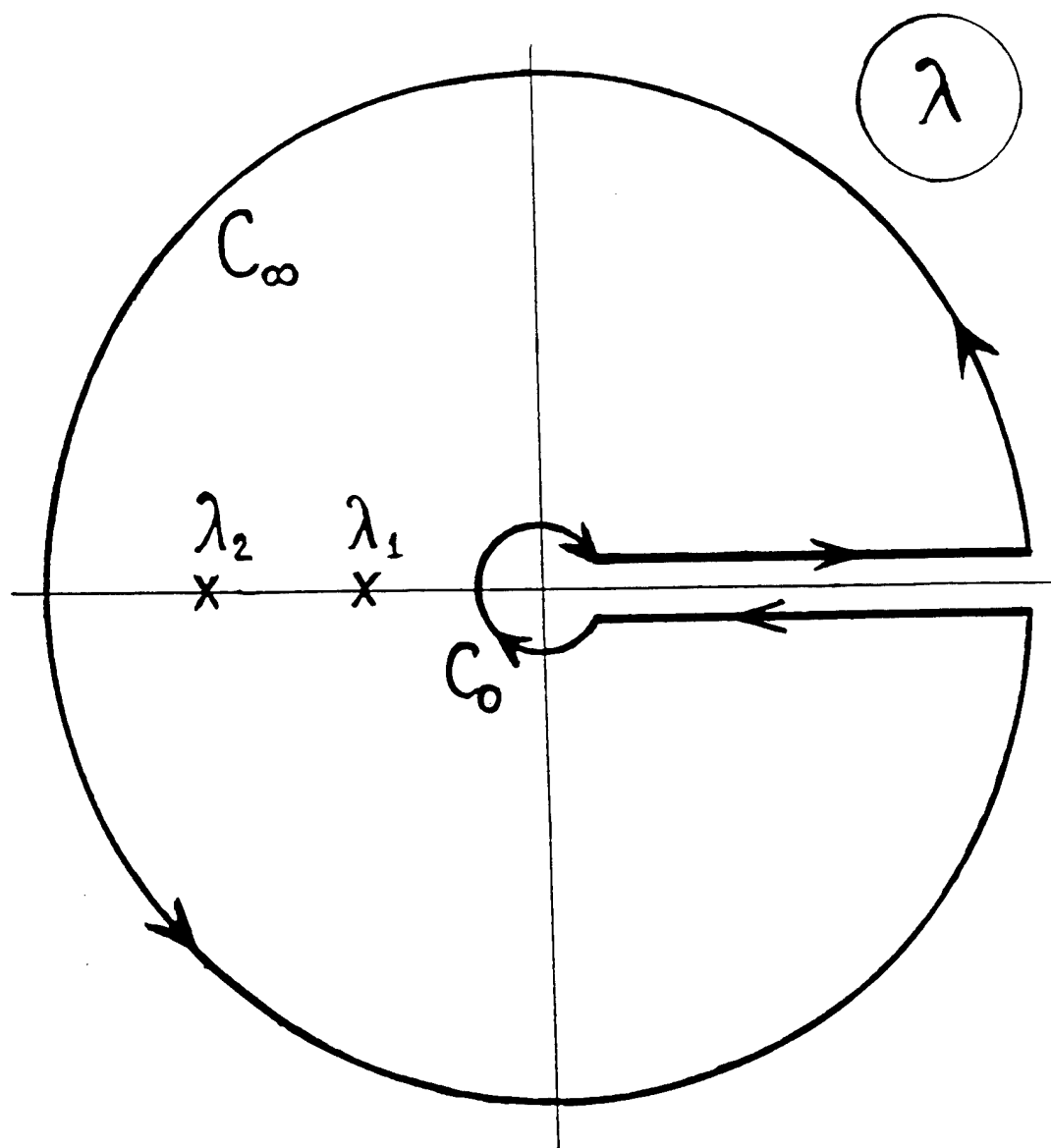Fig.1 Contour used in the derivation of Eq. (4.4).

Fig. 1